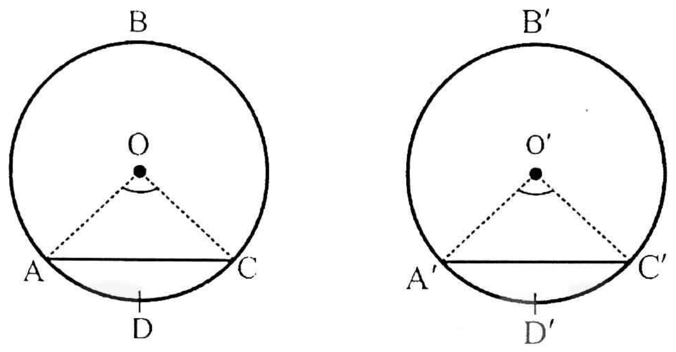
Unit 11

CHORDS AND ARCS

THEOREM 1

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O′ respectively. So that $m\overparen{ADC} = m\overparen{A'D'C'}$

To Prove: $m\overline{AC} = m\overline{A'C'}$

Construction: Join O with A and C, and join O′ with A' and C'.

So that we can form Δ^s OAC and $O'A'C'$.

Proof:

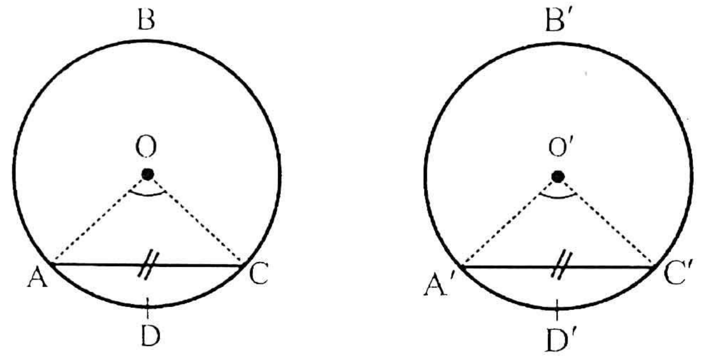
Statements	Reasons
In two equal circles ABCD and $A'B'C'D'$ with centres O and O' respectively.	Given
$m\overparen{ADC} = m\overparen{A'D'C'}$	Given
$\therefore$ $m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.A.S postulate
and in particular $m\overline{AC} = m\overline{A'C'}$	corresponding sides of congruent triangles.
Similarly we can prove the theorem in the same circle.	

THEOREM 2

Converse of Theorem 1

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent. **OR**

In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O′ respectively.

So that chord $m\overline{AC} = m\overline{A'C'}$.

To Prove: $m\widehat{ADC} = m\widehat{A'D'C'}$

Construction: Join O with A and C, and join O′ with A′ and C′.

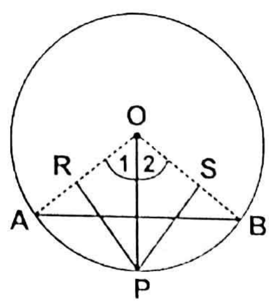
Proof:

Statements	Reasons
In $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\overline{AC} = m\overline{A'C'}$	Given
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.S.S ≅ S.S.S.
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$	Corresponding angles of congruent triangles.
Hence $m\widehat{ADC} = m\widehat{A'D'C'}$	Arcs corresponding to equal central angles.

Example 1: **A point P on the circumference is equidistant from the radii $\overline{OA}$ and $\overline{OB}$. Prove that $m\widehat{AP} = m\widehat{BP}$.**

Given: $\overline{AB}$ is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA}$ and $\overline{OB}$.

So that $m\overline{PR} = m\overline{PS}$.



To Prove: $m\widehat{AP} = m\widehat{BP}$

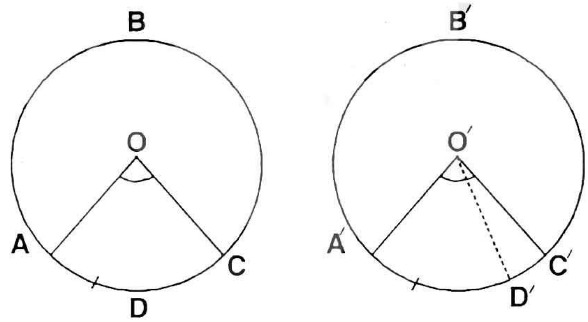
Construction: Join O with P. Write ∠1 and ∠2 as shown in the figure.

Proof:

Statements	Reasons
In $\angle$rt ΔOPR and $\angle$rt ΔOPS	
$m\overline{OP} = m\overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	Point P is equidistance from radii (Given)
$\therefore$ $\Delta OPR \cong \Delta OPS$	(In $\angle$rtΔ^s H.S $\cong$ H.S)
So $m\angle 1 = m\angle 2$	Central angles of a circle
Chord AP $\cong$ Chord BP	
Hence $m\widehat{AP} = m\widehat{BP}$	Arcs corresponding to equal chords in a circle.

THEOREM 3

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).



Given: ABC and $A'B'C'$ are two congruent circles with centres O and O' respectively. So that $m\overline{AC} = m\overline{A'C'}$.

To Prove: $\angle AOC \cong \angle A'O'C'$

Construction: Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

Proof:

Statements	Reasons
$\angle AOC \cong \angle A'O'D'$	Construction
$\therefore$ $\widehat{AC} \cong \widehat{A'D'}$ .... (i)	Arcs subtended by equal Central angles in congruent circles
$\overline{AC} = \overline{A'D'}$ .... (ii)	Using Theorem 1
But $\overline{AC} = \overline{A'C'}$ .... (iii)	Given
$\therefore$ $\overline{A'C'} = \overline{A'D'}$	Using (ii) and (iii)
Which is only possible, if C' coincides with D'.	
Hence $m\angle A'O'C' = m\angle A'O'D'$ .... (iv)	
But $m\angle AOC = m\angle\angle A'O'D'$ ....(v)	Construction
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$	Using (iv) and (v)

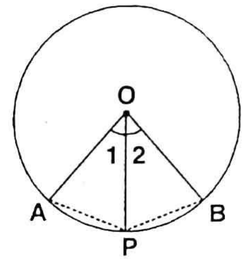
Corollary 1: In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2: In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

Example 1: The internal bisector of a central angle in a circle bisects an arc on which it stands.

Given:

In a circle with centre O. $\overline{OP}$ is an internal bisector of central angle AOB.



To Prove:

$\overset{\frown}{AP} \cong \overset{\frown}{BP}$

Construction:

Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In $\Delta OAP \leftrightarrow \Delta OBP$	
$m\overline{OA} = m\overline{OB}$	Radii of the same circle
$m\angle 1 = m\angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and $m\overline{OP} = m\overline{OP}$	Common
$\Delta OAP \cong \Delta OBP$	S.A.S postulate
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow$ $\overset{\frown}{AP} \cong \overset{\frown}{BP}$	Arcs corresponding to equal chords in a circle.

Example 2: In a circle if any pair of diameters are $\perp$ to each other then the lines joining its ends in order, form a square.

Given:

$\overline{AC}$ and $\overline{BD}$ are two perpendicular diameters of a circle with centre O.

So ABCD is a quadrilateral.

To Prove:

ABCD is a square

Construction:

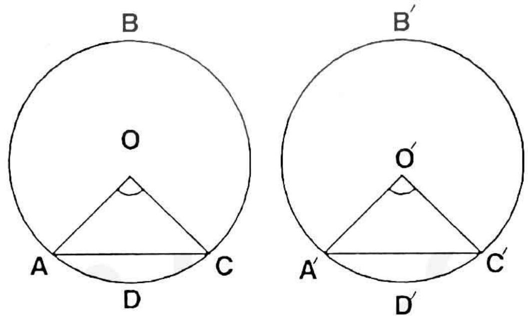
Write $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ as shown in the figure.

Proof:

Statements	Reasons
$\overline{AC}$ and $\overline{BD}$ are two $\perp$ diameters of a circle with centre O.	Given
$\therefore$ m$\angle$1 = m$\angle$2 = m$\angle$3 = m$\angle$4 = 90^{o}	Pair of diameters are $\perp$ to each other.
$\Rightarrow$	Arcs opposite to the equal central angles in a circle.
$\therefore$ $m\widehat{AB} = m\widehat{BC} = m\widehat{CD} = m\widehat{DA}$	Chords corresponding to equal arcs.
$\Rightarrow$ $m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{DA}$ .... (i)	
Moreover m$\angle$A = m$\angle$5 + m$\angle$6	
m$\angle$A = $45^{o} + 45^{0} = 90^{o}$ .... (ii)	Using (i), (ii) and (iii).
Similarly,	
m$\angle$B = m$\angle$C = m$\angle$D = 90^{o} .... (iii)	
Hence ABCD is a square	

THEOREM 4

If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. $\overline{AC}$ and $\overline{A'C'}$ are chords of circles ABCD and $A'B'C'D'$ respectively and m$\angle$AOC= m$\angle A'O'C'$.

To Prove: $m\overline{AC} = m\overline{A'C'}$

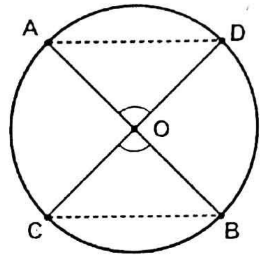
Proof:

Statements	Reasons
In $\Delta OAC \leftrightarrow \Delta O'A'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of congruent circles
m$\angle$AOC = m$\angle A'O'C'$	Given
$m\overline{OC} = m\overline{O'C'}$	Radii of congruent circles
$\therefore$ $\Delta OAC \cong \Delta O'A'C'$ S.A.S	postulate
	Corresponding sides of congruent traingles
Hence $m\overline{AC} = m\overline{A'C'}$	

EXERCISE 11.1

Q.1 In a circle two equal diameters $\overline{AB}$ and $\overline{CD}$ intersect each other. Prove that $m\overline{AD} = m\overline{BC}$.

Given: A circle with centre "O". Two diameters $\overline{AB}$ and $\overline{BC}$, intersecting at point O.



To Prove: $m\overline{AD} = m\overline{BC}$

Construction:

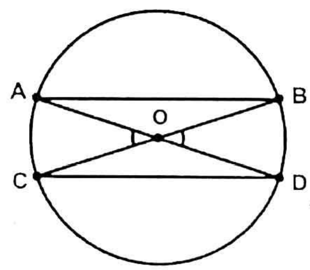
Join A to D and C to B

Proof:

Statements	Reasons
In $\Delta AOD \leftrightarrow \Delta BOC$	
$\overline{OA} \cong \overline{OB}$	Radii of the same circle
$\angle AOD \cong \angle BOC$	Vertical angles are congruent
$\overline{OD} \cong \overline{OC}$	Radii of the same circle
$\therefore$ $\Delta AOD \cong \Delta BOC$	S. A. S ≅ S. A. S
$\overline{AD} \cong \overline{BC}$	Corresponding sides of congruent triangle
Or $m\overline{AD} = m\overline{BC}$	

Q.2. In a circle prove that the arcs between two parallel and equal chords are equal.

Given: A circle with centre O. Two chords $\overline{AB}$ and $\overline{CD}$ Such that $\overline{AB} \parallel \overline{CD}$ and $m\overline{AB} = m\overline{CD}$



To Prove: $m\overset{\frown}{AC} = m\overset{\frown}{BD}$

Construction: Join A to D and B to C. Such that $\overline{AD}$ and $\overline{CD}$ intersect each other at central point O.

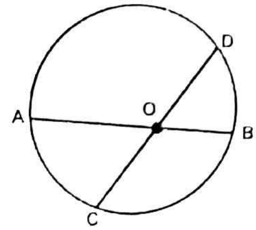
Proof:

Statements	Reasons
$\overline{AD}$ and $\overline{BC}$ are line segment intersecting at centre O.	
$\angle AOC$ and $\angle BOD$ are central angles.	Angle subtended at centre.
$m\angle AOC = m\angle BOD$	Vertical angles
$m\overset{\frown}{AC} = m\overset{\frown}{BB}$	Within the same circle arcs opposite to the equal central angles are equal.

Q.3. Give a geometric proof that a pair of bisecting chords are the diameters of a circle.

Given: A circle and two chords $\overline{AB}$ and $\overline{CD}$ bisecting each other at point O. i.e.

$m\overline{AO} = m\overline{OB}$ and $m\overline{CO} = m\overline{OD}$



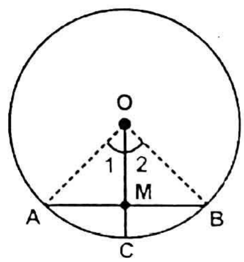
To Prove: Chords $\overline{AB}$ and $\overline{CD}$ are diameters.

Proof:

Statements	Reasons
$m\overline{AB} = m\overline{CD}$ ....(i)	Two chords can bisect each other only when they are equal (given)
$\therefore$ O is the mid point of $\overline{AB}$ and $\overline{CD}$	Given
$m\overline{AO} = m\overline{BO} = \frac{1}{2} m\overline{AB}$ ....(ii)	
$m\overline{DO} = m\overline{CO} = \frac{1}{2} m\overline{CD}$ ....(iii)	
$m\overline{AO} = m\overline{BO} = m\overline{CO} = m\overline{DO}$ ....(iv)	From (i), (ii) and (iii)
The points of circle A, B, C and D are equidistant form the fixed point "O".	From (iv)
This fixed pint O is the centre of the circle having the points A, B, C and D.	By definition
As chords $\overline{AB}$ and $\overline{CD}$ pass through the centre "O" therefore chords $\overline{AB}$ and $\overline{CD}$ are diameters.	

Q.4. If C is the midpoint of an arc ACB in a circle with centre O. Show that line segment OC bisects the chord AB.

Given: A circle with centre "O" $\widehat{ACB}$ is an arc with C as its midpoint and $m\widehat{AC} = m\widehat{CB}$. Center "O" is joined with C such that $\overline{OC}$ meets $\overline{AB}$ at M.



To Prove: $m\overline{AM} = m\overline{BM}$

Construction: Join center "O" with A and B to make central angle AOB.

Proof:

Statements	Reasons
$\angle AOB$ is central angle	Construction
$\therefore \quad m\angle 1 = m\angle 2$ ....(i)	C is the midpoint of $\widehat{ACB}$ (Given)
In $\Delta AOM \longleftrightarrow \Delta BOM$	
$\overline{OM} \cong \overline{OM}$	Common
$\angle 1 \cong \angle 2$	Proved
$\overline{OA} \cong \overline{OB}$	Radii of the same Circle
$\Delta AOM \cong \Delta BOM$	S.A.S ≅ S.A.S
$\overline{AM} \cong \overline{BM}$	Corresponding sides of congruent triangles.
Hence $m\overline{AM} = m\overline{BM}$	

MISCELLANEOUS EXERCISE – 11

Q.1 Multiple Choice Questions

Four possible answers are given for the following questions.

1. A 4 cm long chord subtends central angle of 60°. The radial segment of this, circle
   (a) 1 (b) 2
   (c) 3 (d) 4
2. If an arc of a circle subtends a central angle of 60°, then the corresponding chord of the arc will make the central angle of:
   (a) 20° (b) 40°
   (c) 60° (d) 80°
3. The semi circumference and the diameter of a circle both subtend a central angle of
   (a) 90° (b) 180°
   (c) 270° (d) 360°
4. The arcs opposite to incongruent central angles of a circle arc always:
   (a) Congruent (b) incongruent
   (c) parallel (d) perpendicular
5. If a chord of a circle subtends a central angle of 60°, then the length of the chord and the radial segment are:
   (a) congruent (b) incongruent
   (c) parallel (d) perpendicular
6. The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:
   (a) 30° (b) 45°
   (c) 60° (d) 75°
7. Out of two congruent arcs of a circle, if one arc makes a central angle of 30° then the other arc will subtend the central angle of:
   (a) 15° (b) 30°
   (c) 45° (d) 60°
8. The chord length of a circle subtending a central angle of 180° is always:
   (a) less than radial segment
   (b) equal to the radial segment
   (c) double of the radial segment
   (d) none of these
9. A pair of chords of a circle subtending two congruent central angles is:
   (a) congruent (b) incongruent
   (c) over lapping (d) parallel
10. An arc subtends a central angle of 40° then the corresponding chord will subtended a central angle of:
    (a) 20° (b) 40°
    (c) 60° (d) 80°

ANSWER KEY

1.	d	**2.**	c	**3.**	b	**4.**	b	**5.**	a
6.	c	**7.**	b	**8.**	c	**9.**	a	**10.**	b