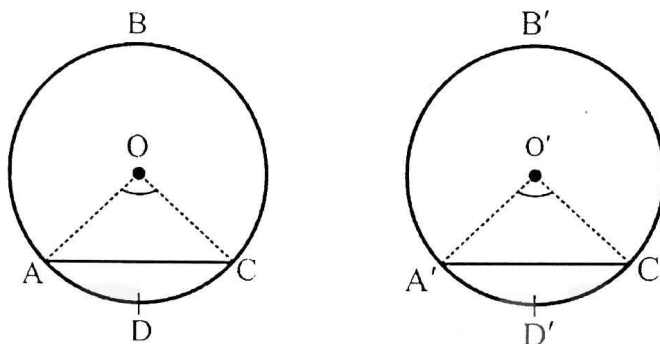


CHORDS AND ARCS

THEOREM 1

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $\widehat{ADC} = \widehat{A'D'C'}$

To Prove: $\widehat{AC} = \widehat{A'C'}$

Construction: Join O with A and C, and join O' with A' and C' .
So that we can form $\triangle OAC$ and $\triangle O'A'C'$.

Proof:

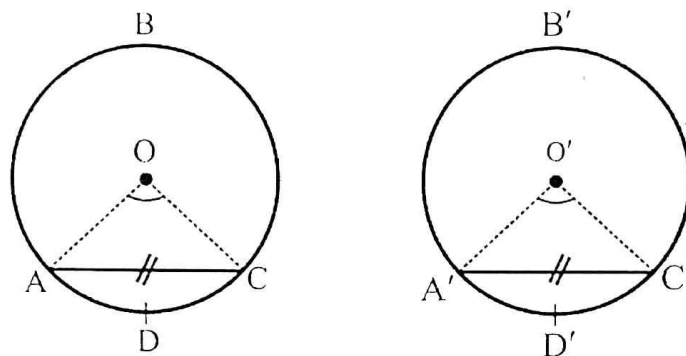
Statements	Reasons
In two equal circles ABCD and $A'B'C'D'$ with centres O and O' respectively.	Given
$\widehat{ADC} = \widehat{A'D'C'}$	Given
$\therefore \angle AOC = \angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\triangle AOC \leftrightarrow \triangle A'O'C'$	
$\overline{OA} = \overline{O'A'}$	Radii of equal circles
$\angle AOC = \angle A'O'C'$	Already Proved
$\overline{OC} = \overline{O'C'}$	Radii of equal circles
$\therefore \triangle AOC \cong \triangle A'O'C'$	S.A.S postulate
and in particular $\widehat{AC} = \widehat{A'C'}$	corresponding sides of congruent triangles.
Similarly we can prove the theorem in the same circle.	

THEOREM 2

Converse of Theorem 1

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent. OR

In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O' respectively.

So that chord $\overline{mAC} = \overline{mA'C'}$.

To Prove: $\widehat{mADC} = \widehat{mA'D'C'}$

Construction: Join O with A and C, and join O' with A' and C' .

Proof:

Statements	Reasons
In $\triangle AOC \leftrightarrow \triangle A'O'C'$	
$\overline{mOA} = \overline{mO'A'}$	Radii of equal circles
$\overline{mOC} = \overline{mO'C'}$	Radii of equal circles
$\overline{mAC} = \overline{mA'C'}$	Given
$\therefore \triangle AOC \cong \triangle A'O'C'$	S.S.S $\cong$ S.S.S.
$\Rightarrow m\angle AOC = m\angle A'O'C'$	Corresponding angles of congruent triangles.
Hence $\widehat{mADC} = \widehat{mA'D'C'}$	Arcs corresponding to equal central angles.

Example 1: A point P on the circumference is equidistant from the radii $\overline{OA}$ and $\overline{OB}$.

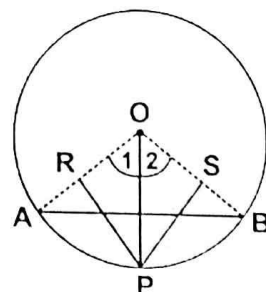
Prove that $\widehat{mAP} = \widehat{mBP}$.

Given: $\overline{AB}$ is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA}$ and $\overline{OB}$.

So that $\overline{mPR} = \overline{mPS}$.

To Prove: $\widehat{mAP} = \widehat{mBP}$

Construction: Join O with P. Write $\angle 1$ and $\angle 2$ as shown in the figure.

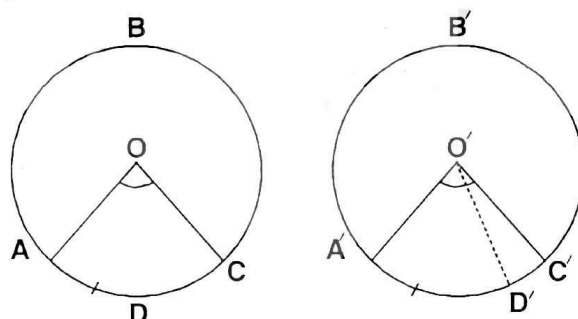


Proof:

Statements	Reasons
In $\angle \text{rt } \triangle OPR$ and $\angle \text{rt } \triangle OPS$ $m\overline{OP} = m\overline{OP}$ $m\overline{PR} = m\overline{PS}$	Common Point P is equidistance from radii (Given)
$\therefore \triangle OPR \cong \triangle OPS$	(In $\angle \text{rt} \Delta^s$ H.S $\cong$ H.S)
So $m\angle 1 = m\angle 2$ Chord $AP \cong$ Chord BP	Central angles of a circle
Hence $m\widehat{AP} = m\widehat{BP}$	Arcs corresponding to equal chords in a circle.

THEOREM 3

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).



Given: ABC and $A'B'C'$ are two congruent circles with centres O and O' respectively. So that $m\overline{AC} = m\overline{A'C'}$.

To Prove: $\angle AOC \cong \angle A'O'C'$

Construction: Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

Proof:

Statements	Reasons
$\angle AOC \cong \angle A'O'D'$ $\therefore \widehat{AC} \cong \widehat{A'D'} \dots\dots\dots (i)$	Construction Arcs subtended by equal Central angles in congruent circles
$\overline{AC} = \overline{A'D'} \dots\dots\dots (ii)$	Using Theorem 1
But $\overline{AC} = \overline{A'C'} \dots\dots\dots (iii)$	Given
$\therefore \overline{A'C'} = \overline{A'D'}$ Which is only possible, if C' coincides with D' .	Using (ii) and (iii)
Hence $m\angle A'O'C' = m\angle A'O'D' \dots\dots\dots (iv)$	
But $m\angle AOC = m\angle A'O'D' \dots\dots\dots (v)$	Construction
$\Rightarrow m\angle AOC = m\angle A'O'C'$	Using (iv) and (v)

Corollary 1: In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2: In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

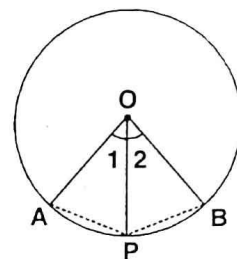
Example 1: The internal bisector of a central angle in a circle bisects an arc on which it stands.

Given:

In a circle with centre O. $\overline{OP}$ is an internal bisector of central angle AOB.

To Prove:

$$\widehat{AP} \cong \widehat{BP}$$



Construction:

Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In $\triangle OAP \leftrightarrow \triangle OBP$	
$\overline{OA} = \overline{OB}$	Radii of the same circle
$m\angle 1 = m\angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and $\overline{OP} = \overline{OP}$	Common
$\triangle OAP \cong \triangle OBP$	S.A.S postulate
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow \widehat{AP} \cong \widehat{BP}$	Arcs corresponding to equal chords in a circle.

Example 2: In a circle if any pair of diameters are $\perp$ to each other then the lines joining its ends in order, form a square.

Given:

$\overline{AC}$ and $\overline{BD}$ are two perpendicular diameters of a circle with centre O.

So ABCD is a quadrilateral.

To Prove:

ABCD is a square

Construction:

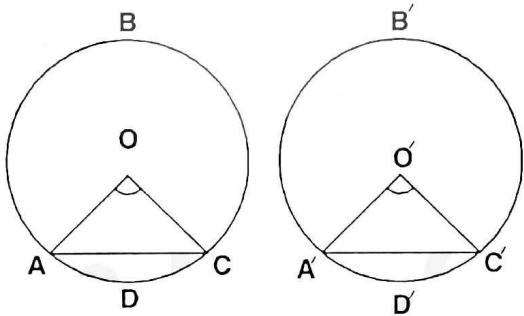
Write $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.

Proof:

Statements	Reasons
$\overline{AC}$ and $\overline{BD}$ are two $\perp$ diameters of a circle with centre O. $\therefore m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^\circ$ $\Rightarrow$ $\therefore m\widehat{AB} = m\widehat{BC} = m\widehat{CD} = m\widehat{DA}$ $\Rightarrow m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{DA}$ (i) Moreover $m\angle A = m\angle 5 + m\angle 6$ $m\angle A = 45^\circ + 45^\circ = 90^\circ$ (ii) Similarly, $m\angle B = m\angle C = m\angle D = 90^\circ$ (iii) Hence ABCD is a square	Given Pair of diameters are $\perp$ to each other. Arcs opposite to the equal central angles in a circle. Chords corresponding to equal arcs. Using (i), (ii) and (iii).

THEOREM 4

If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given: ABCD and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. $\overline{AC}$ and $\overline{A'C'}$ are chords of circles ABCD and $A'B'C'D'$ respectively and $m\angle AOC = m\angle A'O'C'$.

To Prove: $m\overline{AC} = m\overline{A'C'}$

Proof:

Statements	Reasons
In $\triangle OAC \leftrightarrow \triangle O'A'C'$ $m\overline{OA} = m\overline{O'A'}$ $m\angle AOC = m\angle A'O'C'$ $m\overline{OC} = m\overline{O'C'}$ $\therefore \triangle OAC \cong \triangle O'A'C'$ S.A.S	Radii of congruent circles Given Radii of congruent circles postulate Corresponding sides of congruent triangles
Hence $m\overline{AC} = m\overline{A'C'}$	

EXERCISE 11.1

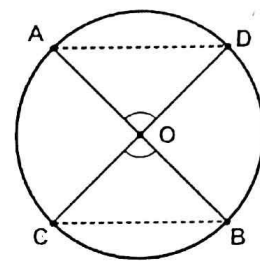
Q.1 In a circle two equal diameters $\overline{AB}$ and $\overline{CD}$ intersect each other. Prove that $m\widehat{AD} = m\widehat{BC}$.

Given: A circle with centre "O". Two diameters $\overline{AB}$ and $\overline{CD}$, intersecting at point O.

To Prove: $m\widehat{AD} = m\widehat{BC}$

Construction:

Join A to D and C to B



Proof:

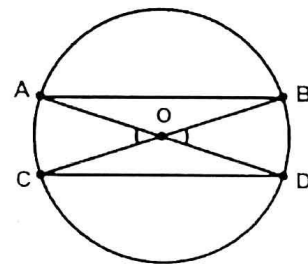
Statements	Reasons
In $\triangle AOD \leftrightarrow \triangle BOC$	
$\overline{OA} \cong \overline{OB}$	Radii of the same circle
$\angle AOD \cong \angle BOC$	Vertical angles are congruent
$\overline{OD} \cong \overline{OC}$	Radii of the same circle
$\therefore \triangle AOD \cong \triangle BOC$	S. A. S $\cong$ S. A. S
$\overline{AD} \cong \overline{BC}$	Corresponding sides of congruent triangle
Or $m\widehat{AD} = m\widehat{BC}$	

Q.2. In a circle prove that the arcs between two parallel and equal chords are equal.

Given: A circle with centre O. Two chords $\overline{AB}$ and $\overline{CD}$ Such that $\overline{AB} \parallel \overline{CD}$ and $m\overline{AB} = m\overline{CD}$

To Prove: $m\widehat{AC} = m\widehat{BD}$

Construction: Join A to D and B to C. Such that $\overline{AD}$ and $\overline{BC}$ intersect each other at central point O.



Proof:

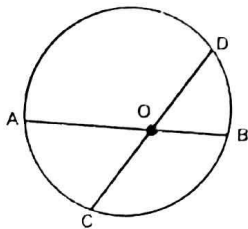
Statements	Reasons
$\overline{AD}$ and $\overline{BC}$ are line segment intersecting at centre O.	
$\angle AOC$ and $\angle BOD$ are central angles.	Angle subtended at centre.
$m\angle AOC = m\angle BOD$	Vertical angles
$m\widehat{AC} = m\widehat{BD}$	Within the same circle arcs opposite to the equal central angles are equal.

Q.3. Give a geometric proof that a pair of bisecting chords are the diameters of a circle.

Given: A circle and two chords $\overline{AB}$ and $\overline{CD}$ bisecting each other at point O. i.e.

$$m\overline{AO} = m\overline{OB} \text{ and } m\overline{CO} = m\overline{OD}$$

To Prove: Chords $\overline{AB}$ and $\overline{CD}$ are diameters.



Proof:

Statements	Reasons
$m\overline{AB} = m\overline{CD}$(i)	Two chords can bisect each other only when they are equal (given) Given
$\therefore$ O is the mid point of $\overline{AB}$ and $\overline{CD}$	
$m\overline{AO} = m\overline{BO} = \frac{1}{2} m\overline{AB}$(ii)	From (i), (ii) and (iii) From (iv) By definition
$m\overline{DO} = m\overline{CO} = \frac{1}{2} m\overline{CD}$(iii)	
$m\overline{AO} = m\overline{BO} = m\overline{CO} = m\overline{DO}$(iv)	
The points of circle A, B, C and D are equidistant from the fixed point "O". This fixed pint O is the centre of the circle having the points A, B, C and D. As chords $\overline{AB}$ and $\overline{CD}$ pass through the centre "O" therefore chords $\overline{AB}$ and $\overline{CD}$ are diameters.	

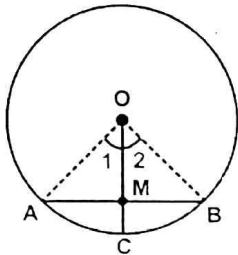
Q.4. If C is the midpoint of an arc ACB in a circle with centre O. Show that line segment OC bisects the chord AB.

Given: A circle with centre "O" $\widehat{ACB}$ is an arc with C as its midpoint and $m\widehat{AC} = m\widehat{CB}$. Center "O" is joined with C such that $\overline{OC}$ meets $\overline{AB}$ at M.

To Prove: $m\overline{AM} = m\overline{BM}$

Construction: Join center "O" with A and B to make central angle AOB.

Proof:



Statements	Reasons
$\angle AOB$ is central angle $\therefore m\angle 1 = m\angle 2$(i)	Construction C is the midpoint of $\widehat{ACB}$ (Given)
In $\triangle AOM \longleftrightarrow \triangle BOM$ $\overline{OM} \cong \overline{OM}$ $\angle 1 \cong \angle 2$ $\overline{OA} \cong \overline{OB}$ $\triangle AOM \cong \triangle BOM$ $\overline{AM} \cong \overline{BM}$ Hence $m\overline{AM} = m\overline{BM}$	Common Proved Radii of the same Circle S.A.S $\cong$ S.A.S Corresponding sides of congruent triangles.