

ANGLE IN A SEGMENT OF A CIRCLE

THEOREM 1

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.

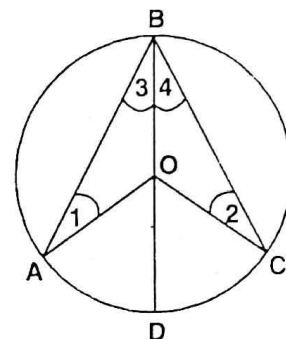
Given: $\widehat{AC}$ is an arc of a circle with centre O

Whereas $\angle AOC$ is the central angle and $\angle ABC$ is circum angle.

To Prove: $m\angle AOC = 2m\angle ABC$

Construction: Join B with O and produce it to meet the circle at D.

Write angles $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.



Proof:

Statements	Reasons
As $m\angle 1 = m\angle 3$ (i)	Angles opposite to equal sides in $\triangle OAB$
and $m\angle 2 = m\angle 4$ (ii)	Angles opposite to equal sides in $\triangle OBC$.
Now $m\angle 5 = m\angle 1 + m\angle 3$ (iii)	External angle is the sum of internal opposite angles.
Similarly $m\angle 6 = m\angle 2 + m\angle 4$ (iv)	
Again $m\angle 5 = m\angle 3 + m\angle 3 = 2m\angle 3$ (v)	Using (i) and (iii)
and $m\angle 6 = m\angle 4 + m\angle 4 = 2m\angle 4$ (vi)	Using (ii) and (iv)
Then from figure	
$\Rightarrow m\angle 5 + m\angle 6 = 2m\angle 3 + 2m\angle 4$	Adding (v) and (vi)
$\Rightarrow m\angle AOC = 2(m\angle 3 + m\angle 4) = 2m\angle ABC$	

Example:

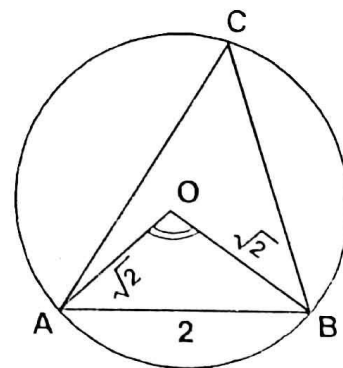
The radius of a circle is $\sqrt{2}$ cm. A chord 2 cm in length divides the circle into two segments. Prove that the angle of larger segment is 45° .

Given: In a circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm.

The length of chord $\overline{AB} = 2$ cm divides the circle into two segments with ACB as larger one.

To Prove: $m\angle ACB = 45^\circ$

Construction: Join O with A and O with B.



Proof:

Statements	Reasons
<p>In $\triangle OAB$</p> $(m\overline{OA})^2 + (m\overline{OB})^2 = (\sqrt{2})^2 + (\sqrt{2})^2$ $= 2 + 2 = 4$ $= (2)^2 = (m\overline{AB})^2$ <p>$\therefore \triangle AOB$ is right angled triangle with $m\angle AOB = 90^\circ$</p> <p>Then $m\angle ACB = \frac{1}{2} m\angle AOB$</p> $= \frac{1}{2} (90^\circ) = 45^\circ$	<p>$m\overline{OA} = m\overline{OB} = \sqrt{2} \text{ cm}$</p> <p>Given: $m\overline{AB} = 2 \text{ cm}$</p> <p>Which being a central angle standing on an arc AB.</p> <p>By theorem 1</p> <p>Circum angle is half of the central angle.</p>

THEOREM 2

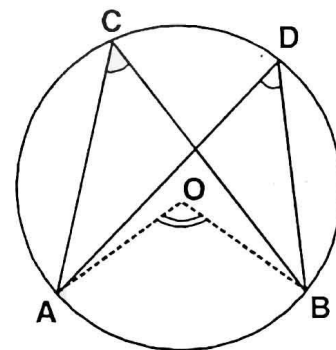
Any two angles in the same segment of a circle are equal.

Given: $\angle ACB$ and $\angle ADB$ are the circum angles in the same segment of a circle with centre O.

To Prove: $m\angle ACB = m\angle ADB$

Construction: Join O with A and O with B.

So that $\angle AOB$ is the central angle.



Proof:

Statements	Reasons
<p>Standing on the same arc AB of a circle.</p> <p>$\angle AOB$ is the central angle whereas</p> <p>$\angle ACB$ and $\angle ADB$ are circum angles</p> <p>$\therefore m\angle AOB = 2m\angle ACB$ (i)</p> <p>and $m\angle AOB = 2m\angle ADB$ (ii)</p> <p>$\Rightarrow 2m\angle ACB = 2m\angle ADB$</p> <p>Hence, $m\angle ACB = m\angle ADB$</p>	<p>Construction</p> <p>Given</p> <p>By theorem 1</p> <p>By theorem 1</p> <p>Using (i) and (ii)</p>

THEOREM 3

The angle,

- In a semi-circle is a right angle,
- In a segment greater than a semi circle is less than a right angle,
- In a segment less than a semi-circle is greater than a right angle.

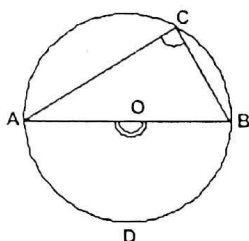


Fig. I

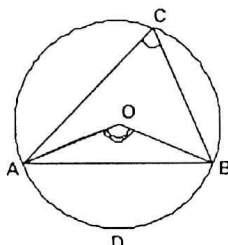


Fig. II

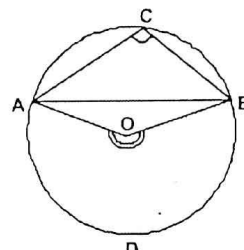


Fig. III

Given: $\overline{AB}$ is the chord corresponding to an arc ADB

Whereas $\angle AOB$ is a central angle and $\angle ACB$ is a circum angle of a circle with centre O.

To Prove:

In fig (I) If sector ACB is a semi circle then $m\angle ACB = \frac{1}{2} \angle rt$

In fig (II) If sector ACB is greater than a semi circle then $m\angle ACB < \frac{1}{2} \angle rt$

In fig (III) If sector ACB is less than a semi circle then $m\angle ACB > \frac{1}{2} \angle rt$.

Proof:

Statements	Reasons
In each figure, $\overline{AB}$ is the chord of a circle with centre O.	Given
$\angle AOB$ is the central angle standing on an arc ADB.	Given
Whereas $\angle ACB$ is the circum angle	
Such that $m\angle AOB = 2m\angle ACB$ (i)	
Now in fig (I) $m\angle AOB = 180^\circ$	By theorem 1
$\therefore m\angle AOB = 2\angle rt$ (ii)	A straight angle
$\Rightarrow m\angle ACB = \frac{1}{2} \angle rt$	
In fig (II) $m\angle AOB < 180^\circ$	Using (i) and (ii)
$\therefore m\angle AOB < 2\angle rt$ (iii)	
$\Rightarrow m\angle ACB < \frac{1}{2} \angle rt$	
In fig (III) $m\angle AOB > 180^\circ$	Using (i) and (iii)
$\therefore m\angle AOB > 2\angle rt$ (iv)	
$\Rightarrow m\angle ACB > \frac{1}{2} \angle rt$	Using (i) and (iv)

Corollary 1:

The angles subtended by an arc at the circumference of a circle are equal.

Corollary 2:

The angles in the same segment of a circle are congruent.

THEOREM 4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

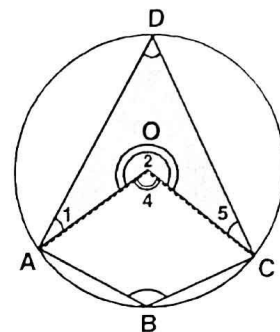
Given: ABCD is a quadrilateral inscribed in a circle with centre O.

To Prove: $\begin{cases} m\angle A + m\angle C = 2\angle rts \\ m\angle B + m\angle D = 2\angle rts \end{cases}$

Construction: Draw $\overline{OA}$ and $\overline{OC}$.

Write $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and

$\angle 6$ as shown in the figure.



Proof:

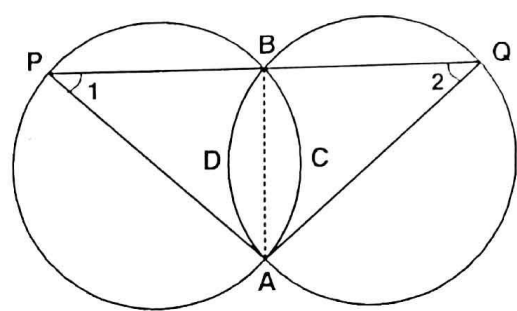
Statements	Reasons
Standing on the same arc ADC, $\angle 2$ is a central angle. Whereas $\angle B$ is the circum angle	Arc ADC of the circle with centre O.
$\therefore m\angle B = \frac{1}{2} (m\angle 2)$ (i)	By theorem 1
Standing on the same arc ABC, $\angle 4$ is a central angle whereas $\angle D$ is the circum angle.	Arc ABC of the circle with centre O.
$\therefore m\angle D = \frac{1}{2} (m\angle 4)$ (ii)	By theorem 1
$\Rightarrow m\angle B + m\angle D = \frac{1}{2} m\angle 2 + \frac{1}{2} m\angle 4$ $= \frac{1}{2} (m\angle 2 + m\angle 4) = \frac{1}{2} (\text{Total central angle})$ i.e., $m\angle B + m\angle D = \frac{1}{2} (4\angle rts) = 2\angle rts$ Similarly $m\angle A + m\angle C = 2\angle rts$	Adding (i) and (ii)

Corollary 1: In equal circles or in the same circle if two minor arcs are equal then angles inscribed by their corresponding major arcs are also equal.

Corollary 2: In equal circles or in the same circle, two equal arcs subtend equal angles at the circumference and vice versa.

Example 1: Two equal circles intersect in A and B. Through B, a straight line is drawn to meet the circumferences at P and Q respectively. Prove that $m\overline{AP} = m\overline{AQ}$.

Given: Two equal circles cut each other at points A and B. A straight line PBQ drawn through B meets the circles at P and Q respectively.



To Prove: $m\overline{AP} = m\overline{AQ}$

Construction: Join the points A and B. Also draw $\overline{AP}$ and $\overline{AQ}$.

Write $\angle 1$ and $\angle 2$ as shown in the figure.

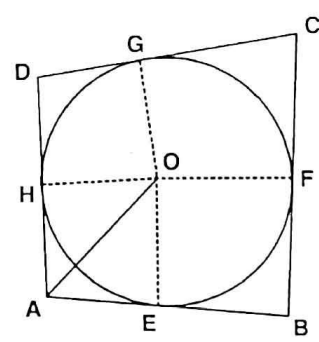
Proof

Statements	Reasons
$\therefore m\widehat{ACB} = m\widehat{ADB}$	Arcs about the common chord AB. Corresponding angles made by opposite arcs. Sides opposite to equal angles in $\triangle APQ$.
$\therefore m\angle 1 = m\angle 2$	
So $m\overline{AQ} = m\overline{AP}$	
or $m\overline{AP} = m\overline{AQ}$	

Example 2: ABCD is a quadrilateral circumscribed about a circle.

Show that $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$

Given: ABCD is a quadrilateral circumscribed about a circle with centre O. So that each side becomes tangent to the circle.



To Prove: $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$

Construction: Drawn $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{BC}$

$\overline{OG} \perp \overline{CD}$ and $\overline{OH} \perp \overline{DA}$

Proof

Statements	Reasons
$\therefore m\overline{AE} = m\overline{HA}$ and $m\overline{EB} = m\overline{BF}$(i)	Since tangents drawn from a point to the circle are equal in length. Adding (i) and (ii).
$m\overline{CG} = m\overline{FC}$ and $m\overline{GD} = m\overline{DH}$(ii)	
$(m\overline{AE} + m\overline{EB}) + (m\overline{CG} + m\overline{GD}) = (m\overline{BF} + m\overline{FC}) + (m\overline{DH} + m\overline{HA})$	
or $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$	

EXERCISE 12.1

Q.1 Prove that in a given cyclic quadrilateral, sum of opposite angles is two right angles and conversely.

Given: A circle with centre "O"

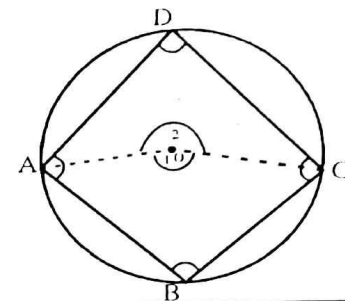
ABCD is a cyclic quadrilateral

To Prove: $m\angle B + m\angle D = 180^\circ$

$$m\angle BCD + m\angle DAB = 180^\circ$$

Construction: Join O with A and C

Proof:



Statements	Reasons
$m\angle 1 = 2m\angle D \dots\dots\dots (i)$ $m\angle 2 = 2m\angle B \dots\dots\dots (ii)$ $m\angle 1 + m\angle 2 = 2m\angle D + 2m\angle B$ $m\angle 1 + m\angle 2 = 2(m\angle D + m\angle B)$ or $2(m\angle D + m\angle B) = m\angle 1 + m\angle 2$ $2(m\angle D + m\angle B) = 360^\circ$ $m\angle D + m\angle B = \frac{360^\circ}{2}$ $m\angle D + m\angle B = 180^\circ$ Similarly $m\angle BCD + m\angle DAB = 180^\circ$	$\angle 1, \angle 2$ are central angles and $\angle D, \angle B$ are circum angles in Arcs Adding (i) and (ii) By symmetric property Sum of all central angles is 360° Dividing by 2

Q.2 Show that parallelogram inscribed in a circle will be a rectangle.

Given: ABCD is a parallelogram inscribed in the circle with centre "O"

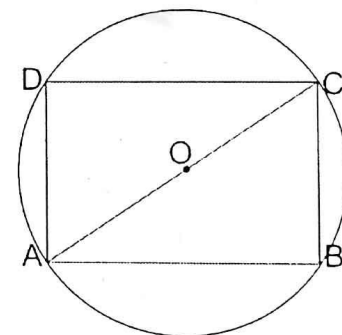
$$m\overline{AB} = m\overline{DC} \text{ and } \overline{AB} \parallel \overline{DC}$$

$$m\overline{AD} = m\overline{BC} \text{ and } \overline{AD} \parallel \overline{BC}$$

To Prove: ABCD is a rectangle

Construction: Join A with C

Proof:



Statements	Reasons
In $\triangle ABC \leftrightarrow \triangle ADC$ $m\overline{AC} = m\overline{AC}$ $m\overline{AB} = m\overline{DC}$ $m\overline{BC} = m\overline{AD}$ $\therefore \triangle ABC \cong \triangle ADC$ Thus, $m\angle B = m\angle D \dots\dots\dots (i)$ $m\angle B + m\angle D = 180^\circ \dots\dots\dots (ii)$ $\Rightarrow m\angle B = m\angle D = 90^\circ$ Similarly $m\angle BAD = m\angle BCD = 90^\circ$ Hence ABCD is rectangle	Common Given Given S.S. S $\cong$ S. S. S Corresponding angles of congruent triangles Opposite angles of parallelogram From (i) From (i) and (ii)

Q.3 $\overline{AOB}$ and $\overline{COD}$ are two intersecting chords of a circle.

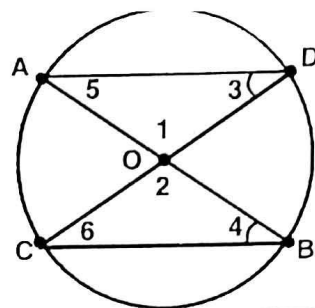
Show that $\triangle AOD$ and $\triangle BOC$ are equiangular.

Given: In a circle $\overline{AOB}$ and $\overline{COD}$ are two intersecting chords at point O.

To Prove: $\triangle AOD$ and $\triangle BOC$ are equiangular

Construction: Join A with C and D. Join B with C and D.

Proof:



Statements	Reasons
$m\angle 1 \cong m\angle 2$(i)	Vertical angles
$\overline{AC}$ is chord and angles $\angle 3, \angle 4$ are in the same segment. $\angle 3 \cong \angle 4$(ii)	
Now $\overline{BD}$ is chord and angles $\angle 5, \angle 6$ are in the same segments	From (i), (ii) and (iii)
Therefore $\angle 5 \cong \angle 6$(iii)	
Thus, $\triangle AOD$ and $\triangle BOC$ are equiangular	

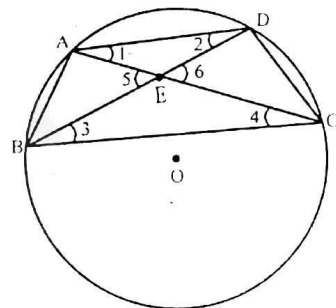
Q.4 $\overline{AD}$ and $\overline{BC}$ are two parallel chords of a circle prove that arc $\widehat{AB} \cong$ arc $\widehat{CD}$ and arc $\widehat{AC} \cong$ arc $\widehat{BD}$.

Given: A circle with centre "O". Two chords $\overline{AD}$ and $\overline{BC}$ are such that $\overline{AD} \parallel \overline{BC}$.

To Prove: arc $\widehat{AB} \cong$ arc $\widehat{CD}$ and arc $\widehat{AC} \cong$ arc $\widehat{BD}$

Construction: Join A to B and C. Join D to B and C. $\overline{AC}$ and $\overline{BD}$ intersect each other at point E. some angles are named as $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5, \angle 6$.

Proof:



Statements	Reasons
$m\angle 1 = m\angle 3$(i)	Angles inscribed by an arc in the same segment are equal.
$m\angle 2 = m\angle 4$(ii)	
$m\angle 1 = m\angle 4$(iii)	Alternate angles are congruent ($\overline{AD} \parallel \overline{BC}$)
$m\angle 3 = m\angle 4$(iv)	
$m\angle 1 = m\angle 2$(v)	From (i) and (iii)
In $\triangle AEB \leftrightarrow \triangle DEC$	From (ii) and (iii)
$\overline{AE} \cong \overline{ED}$	Side opposite to equal angles (v)
$m\angle 5 = m\angle 6$	vertical angles
$\overline{BE} \cong \overline{EC}$	Sides opposite to equal angles (iv)
$\therefore \triangle AED \cong \triangle DEC$	S.A.S $\cong$ S.A.S
$\overline{AB} \cong \overline{CD}$	Corresponding sides of congruent.
Thus arc $\widehat{AB} \cong$ arc $\widehat{CD}$ (Hence Proved)	Arcs corresponding to congruent chords are congruent.
$\widehat{mBC} \cong \widehat{mCB}$	Self congruent
$\widehat{mBA} + \widehat{mAC} = \widehat{mCD} + \widehat{mDB}$	
$\widehat{mAB} + \widehat{mAC} = \widehat{mAB} + \widehat{mBD}$	$\therefore$ arc $\widehat{AB} \cong$ arc $\widehat{CD}$ proved
$\widehat{mAC} = \widehat{mBD}$	
or arc $\widehat{AC} \cong$ arc $\widehat{BD}$ (Hence proved)	