

Nature of the roots of a quadratic equation:

On solving quadratic equations, we get different kinds of roots. Now we will discuss the nature or characteristics of the roots of the quadratic equation without actually solving it.

Discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$

We know that two roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$

$$\text{Are } \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The nature of these roots depends on the value of the expression " $b^2 - 4ac$ " which is called the "**discriminant**" of the quadratic equation or the quadratic expression $ax^2 + bx + c$.

To find the discriminant of a given quadratic equation.

Example 1: Find the discriminant of the following equation.

(a) $2x^2 - 7x + 1 = 0$ (b) $x^2 - 3x + 3 = 0$

Solution:

(a) $2x^2 - 7x + 1 = 0$

Here $a = 2$, $b = -7$, $c = 1$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-7)^2 - 4(2)(1) \\ &= 49 - 8 = 41 \end{aligned}$$

(b) $x^2 - 3x + 3 = 0$

Here $a = 1$, $b = -3$, $c = 3$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(3) \\ &= 9 - 12 = -3 \end{aligned}$$

Nature of the roots of a quadratic equation through discriminant.

The roots of the quadratic equation $ax^2 + bx + c = 0$,

($a \neq 0$) are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and its

discriminant is $b^2 - 4ac$.

- i. If $b^2 - 4ac > 0$ and is a perfect square, then the roots are rational (real) and unequal.
- ii. If $b^2 - 4ac > 0$ and is not a perfect square, then the roots are irrational (real) and unequal.
- iii. If $b^2 - 4ac = 0$, then the roots are rational (real) and equal.
- iv. If $b^2 - 4ac < 0$, then the roots are imaginary (complex conjugates).

Determine the nature of the roots of a given quadratic equation and verify the result by solving the equation.

Example 2: Using discriminant, find the nature of the roots of the following quadratic equations and verify the results by solving the equations.

(a) $x^2 - 5x + 5 = 0$ (b) $2x^2 - x + 1 = 0$

(c) $x^2 + 8x + 16 = 0$ (d) $7x^2 + 8x + 1 = 0$

Solution: (a) $x^2 - 5x + 5 = 0$

Compare with the standard quadratic equation $ax^2 + bx + c = 0$

Here $a = 1$, $b = -5$ and $c = 5$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-5)^2 - 4(1)(5) \\ &= 25 - 20 \\ &= 5 > 0 \end{aligned}$$

As the Disc. is positive and is not a perfect square. Therefore the roots are (real) irrational and unequal. Now solving the equation

$x^2 - 5x + 5 = 0$ by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

Evidently, the roots are (real) irrational and unequal.

(b) $2x^2 - x + 1 = 0$

Here $a = 2$, $b = -1$ and $c = 1$

Disc. $= b^2 - 4ac$
 $= (-1)^2 - 4(2)(1) = 1 - 8 = -7 < 0$

As the disc. is negative, therefore the roots of the equation are imaginary and unequal.

Verification by solving the equation.

$$2x^2 - x + 1 = 0$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{1 - 8}}{4} = \frac{1 \pm \sqrt{-7}}{4}$$

$$x = \frac{1 \pm i\sqrt{7}}{4}$$

Evidently, the roots are imaginary and unequal.

(c) $x^2 + 8x + 16 = 0$

Here $a = 1$, $b = 8$ and $c = 16$

Disc. $= b^2 - 4ac$
 $= (8)^2 - 4(1)(16)$
 $= 64 - 64 = 0$

As the discriminant is zero, therefore the roots are (real) rational and equal.

Verification by solving the equation.

$$x^2 + 8x + 16 = 0$$

$$x^2 + 4x + 4x + 16 = 0$$

$$x(x+4) + 4(x+4) = 0$$

$$(x+4)(x+4) = 0$$

$$x+4 = 0 \text{ or } x+4 = 0$$

$$\Rightarrow x = -4 \text{ or } x = -4$$

so the roots are (real) rational and equal.

(d) $7x^2 + 8x + 1 = 0$

Here $a = 7$, $b = 8$ and $c = 1$

Disc. $= b^2 - 4ac$
 $= (8)^2 - 4(7)(1)$
 $= 64 - 28 = 36 = (6)^2$

Which is positive and perfect square.

$\therefore$ The roots are (real) rational and unequal.

Now solving the equation by factorization we get

$$7x^2 + 8x + 1 = 0$$

$$7x^2 + 7x + x + 1 = 0$$

$$7x(x+1) + 1(x+1) = 0$$

$$(x+1)(7x+1) = 0$$

Either $x+1 = 0$ or $7x+1 = 0$, that is

$$x = -1 \text{ or } 7x = -1 \Rightarrow x = -\frac{1}{7}$$

Thus, the roots are (real) rational and unequal.

To determine the value of an unknown involved in a given quadratic equation when nature of its roots is given.

Examples 3:

Find k , if the roots of the equation

$$(k+3)x^2 - 2(k+1)x - (k+1) = 0 \text{ are equal, if } k \neq -3.$$

Solution:

$$(k+3)x^2 - 2(k+1)x - (k+1) = 0$$

Here $a = k+3$, $b = -2(k+1)$ and $c = -(k+1)$

$\therefore$ As roots are equal, so Disc. $= 0$, that is,

$$\therefore b^2 - 4ac = 0$$

$$[-2(k+1)]^2 - 4(k+3)[-(k+1)] = 0$$

$$4(k+1)^2 + 4(k+3)(k+1) = 0$$

$$4(k+1)[(k+1) + (k+3)] = 0$$

$$\text{or } 4(k+1)[k+1+k+3] = 0$$

$$\Rightarrow 4(k+1)(2k+4) = 0$$

$$\Rightarrow 4(k+1)2(k+2) = 0$$

$$\text{or } 8(k+1)(k+2) = 0$$

Since $8 \neq 0$, so

$$\Rightarrow k+1 = 0 \text{ or } k+2 = 0$$

$$\Rightarrow k = -1 \text{ or } k = -2$$

Thus, roots will be equal if $k = -2, -1$.

EXERCISE 2.1

Q.1 Find the discriminant of the following given quadratic equation.

Solutions:

(i) $2x^2 + 3x - 1 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 2, b = 3, c = -1$

Disc. $= b^2 - 4ac$

$= (3)^2 - 4(2)(-1)$

$= 9 + 8$

$= 17$

(ii) $6x^2 - 8x + 3 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 6, b = -8, c = 3$

Disc. $= b^2 - 4ac$

$= (-8)^2 - 4(6)(3)$

$= 64 - 72$

$= -8$

(iii) $9x^2 - 30x + 25 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 9, b = -30, c = 25$

Dis $= b^2 - 4ac$

$= (-30)^2 - 4(9)(25)$

$= 900 - 900$

$= 0$

(iv) $4x^2 - 7x - 2 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 4, b = -7, c = -2$

Disc $= b^2 - 4ac$

$= (-7)^2 - 4(4)(-2)$

$= 49 + 32$

$= 81$

Q.2 Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations

(i) $x^2 - 23x + 120 = 0$

Solutions:

$x^2 - 23x + 120 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 1, b = -23, c = 120$

Disc $= b^2 - 4ac$

$= (-23)^2 - 4(1)(120)$

$= 529 - 480$

$= 49$

$= (7)^2$

As Disc. is positive and perfect square, therefore roots of equation are real, rational and unequal.

Verification.

Now solving the Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-23) \pm \sqrt{(-23)^2 - 4(1)(120)}}{2(1)}$$

$$x = \frac{23 \pm \sqrt{529 - 480}}{2}$$

$$x = \frac{23 \pm \sqrt{49}}{2}$$

$$x = \frac{23 \pm 7}{2}$$

$$x = \frac{23 + 7}{2}$$

or

$$x = \frac{23 - 7}{2}$$

$$x = \frac{30}{2}$$

or

$$x = \frac{16}{2}$$

$$x = 15$$

or

$$x = 8$$

Thus roots are real, rational and unequal.

(ii) $2x^2 + 3x + 7 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 2, b = 3, c = 7$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(7) \\ &= 9 - 56 \\ &= -47 < 0\end{aligned}$$

As Disc. is negative, therefore roots of the equation are imaginary (complex conjugates).

Verification

We verify the results by solving the equation.

$$2x^2 + 3x + 7 = 0$$

$$a = 2, b = 3, c = 7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9 - 56}}{4}$$

$$x = \frac{-3 \pm \sqrt{-47}}{4}$$

Thus roots are imaginary.

(iii) $16x^2 - 24x + 9 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 16, b = -24, c = 9$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (-24)^2 - 4(16)(9) \\ &= 576 - 576 \\ &= 0\end{aligned}$$

As Disc. = 0 therefore roots of equation are rational (real) and equal.

Verification:

We verify the result by solving the equation.

Here

$$a = 16, b = -24, c = 9$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$x = \frac{24 \pm \sqrt{576 - 576}}{32}$$

$$x = \frac{24 \pm \sqrt{0}}{32}$$

$$x = \frac{24 \pm 0}{32}$$

$$x = \frac{24 + 0}{32} \quad \text{or} \quad x = \frac{24 - 0}{32}$$

$$x = \frac{24}{32} \quad \text{or} \quad x = \frac{24}{32}$$

$$x = \frac{3}{4} \quad \text{or} \quad x = \frac{3}{4}$$

Thus roots are rational (real) and equal.

(iv) $3x^2 + 7x - 13 = 0$

As $ax^2 + bx + c = 0$

$\Rightarrow a = 3, b = 7, c = -13$

$$\begin{aligned}\text{Disc} &= b^2 - 4ac \\ &= (7)^2 - 4(3)(-13) \\ &= 49 + 156 \\ &= 205\end{aligned}$$

Therefore roots of equation are real, irrational, and unequal.

Verification:

We verify the result by solving the equation.

$$3x^2 + 7x - 13 = 0$$

$$a = 3, b = 7, c = -13$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$x = \frac{-7 \pm \sqrt{205}}{6}$$

Thus roots are real, irrational and unequal.

Q.3 For what value of k , the expression $k^2x^2 + 2(k+1)x + 4$ is perfect square.

Solution:

$$a = k^2, \quad b = 2(k+1), \quad c = 4$$

$$\begin{aligned} b^2 - 4ac &= [2(k+1)]^2 - 4(k^2)(4) \\ &= 2^2(k+1)^2 - 16k^2 \\ &= 4(k^2 + 2k + 1) - 16k^2 \\ &= 4k^2 + 8k + 4 - 16k^2 \\ &= -12k^2 + 8k + 4 \end{aligned}$$

As given that expression is perfect square therefore

$$\text{Disc} = 0$$

$$-12k^2 + 8k + 4 = 0$$

$$-4(3k^2 - 2k - 1) = 0$$

$$\Rightarrow 3k^2 - 2k - 1 = 0 \quad (\because -4 \neq 0)$$

$$3k^2 - 3k + k - 1 = 0$$

$$3k(k-1) + 1(k-1) = 0$$

$$(k-1)(3k+1) = 0$$

$$k-1=0 \quad \text{or} \quad 3k+1=0$$

$$k=0+1 \quad \text{or} \quad 3k=0-1$$

$$k=1 \quad \text{or} \quad k=\frac{-1}{3}$$

$$\text{So values of } k \text{ are } 1 \text{ and } \frac{-1}{3}$$

Q.4 Find the value of K , if the roots of the following equations are equal.

Solution:

(i) $(2k-1)x^2 + 3kx + 3 = 0$

As $ax^2 + bx + c = 0$

$$a = 2k-1, \quad b = 3k, \quad c = 3$$

$$\text{Disc.} = b^2 - 4ac$$

$$= (3k)^2 - 4(2k-1)(3)$$

$$= 9k^2 - 12(2k-1)$$

$$= 9k^2 - 24k + 12$$

As the roots of given equation are equal, So

$$\text{Disc.} = 0$$

$$9k^2 - 24k + 12 = 0$$

$$3(3k^2 - 8k + 4) = 0$$

$$3k^2 - 8k + 4 = 0 \quad (\because 3 \neq 0)$$

$$3k^2 - 6k - 2k + 4 = 0$$

$$3k(k-2) - 2(k-2) = 0$$

$$(k-2)(3k-2) = 0$$

$$\text{Either } k-2=0 \quad \text{or} \quad 3k-2=0$$

$$k=2 \quad \text{or} \quad 3k=2$$

$$\text{or} \quad k = \frac{2}{3}$$

$$\text{So values of } k \text{ are } 2 \text{ and } \frac{2}{3}$$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

Solution:

As $ax^2 + bx + c = 0$

$$a = 1, \quad b = 2(k+2), \quad c = 3k+4$$

$$\text{Disc.} = b^2 - 4ac$$

$$= [2(k+2)]^2 - 4(1)(3k+4)$$

$$= 4(k+2)^2 - 4(3k+4)$$

$$= 4(k^2 + 4k + 4) - 12k - 16$$

$$= 4k^2 + 16k + 16 - 12k - 16$$

$$= 4k^2 + 4k$$

$$= 4k(k+1)$$

As roots of given equation are equal, so

$$\text{Disc.} = 0$$

$$4k(k+1) = 0$$

$$\text{Either } 4k=0 \quad \text{or} \quad k+1=0$$

$$\text{As } 4 \neq 0 \quad \text{or} \quad k=-1$$

$$\Rightarrow k=0$$

So values of k are 0 and -1

(iii) $(3k+2)x^2 - 5(k+1)x + (2k+3) = 0$

Solution:

As $ax^2 + bx + c = 0$

$$a = (3k+2), \quad b = -5(k+1), \quad c = 2k+3$$

$$\text{Disc} = b^2 - 4ac$$

$$= [-5(k+1)]^2 - 4(3k+2)(2k+3)$$

$$= (-5)^2(k+1)^2 - 4(6k^2 + 9k + 4k + 6)$$

$$= 25[k^2 + 2(k)(1) + 1^2] - 4(6k^2 + 13k + 6)$$

$$\begin{aligned}
&= 25k^2 + 50k + 25 - 24k^2 - 52k - 24 \\
&= k^2 - 2k + 1 \\
&= k^2 - k - k + 1 \\
&= k(k-1) - 1(k-1) \\
&= (k-1)(k-1)
\end{aligned}$$

As root of given equation are equal

$$\text{Disc} = 0$$

$$(k-1)(k-1) = 0$$

$$k-1=0 \quad \text{or} \quad k-1=0$$

$$k=1 \quad \text{or} \quad k=1$$

So values of k is 1

Q.5 Show that the equation $x^2 + (mx+c)^2 = a^2$ has equal roots if

$$c^2 = a^2(1+m^2)$$

Solution: We have given

$$x^2 + (mx+c)^2 = a^2$$

$$x^2 + (mx)^2 + (c)^2 + 2(mx)(c) = a^2$$

$$x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$$

$$(1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$$

$$Ax^2 + Bx + C = 0$$

$$A = (1+m^2), B = 2mc, C = c^2 - a^2$$

$$\text{Disc} = B^2 - 4AC$$

$$= (2mc)^2 - 4(1+m^2)(c^2 - a^2)$$

$$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - a^2m^2)$$

$$= \cancel{4m^2c^2} - 4c^2 + 4a^2 - \cancel{4m^2c^2} + 4a^2m^2$$

$$= -4c^2 + 4a^2 + 4a^2m^2$$

As roots of the equation are equal,

$$\text{So Disc} = 0$$

$$-4c^2 + 4a^2 + 4a^2m^2 = 0$$

$$-4[c^2 - a^2 - a^2m^2] = 0$$

$$\text{As } -4 \neq 0, \text{ so } c^2 - a^2 - a^2m^2 = 0$$

$$c^2 = a^2 + a^2m^2$$

$$c^2 = a^2(1+m^2)$$

Hence proved

Q.6 Find the condition that the roots of the equation $(mx+c)^2 - 4ax = 0$ are equal

Solution: We have given

$$(mx+c)^2 - 4ax = 0$$

$$(mx^2) + (c)^2 + 2(mx)(c) - 4ax = 0$$

$$m^2x^2 + c^2 + 2mcx - 4ax = 0$$

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

$$Ax^2 + Bx + C = 0$$

$$A = m^2, B = 2mc - 4a, C = c^2$$

$$\text{Disc.} = B^2 - 4AC$$

$$= (2mc - 4a)^2 - 4(m^2)(c^2)$$

$$= (2mc)^2 + (4a)^2 - 2(2mc)(4a) - 4m^2c^2$$

$$= \cancel{4m^2c^2} + 16a^2 - 16acm - \cancel{4m^2c^2}$$

$$= 16a^2 - 16acm$$

$$= 16a(a - mc)$$

As root of the equation are equal,

$$\text{So Disc} = 0$$

$$16a(a - mc) = 0$$

$$\text{Either } 16a = 0 \text{ or } a - mc = 0$$

$$\text{As } 16 \neq 0 \text{ So } a = 0 \text{ or } a = mc$$

Thus required condition is $a=0$ or $a=mc$

Q.7 If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

Solution: We have given

$$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$$

$$Ax^2 + Bx + C = 0$$

$$A = (c^2 - ab), B = -2(a^2 - bc), C = (b^2 - ac)$$

$$\text{Disc.} = B^2 - 4AC$$

$$= [-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac)$$

$$= (-2)^2(a^2 - bc)^2 - 4(c^2 - ab)(b^2 - ac)$$

$$= 4[(a^2)^2 + (bc)^2 - 2(a^2)(bc)] - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$$

$$= 4(a^4 + b^2c^2 - 2a^2bc) - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$$

$$= 4a^4 + \cancel{4b^2c^2} - 8a^2bc - \cancel{4b^2c^2} + 4ac^3 + 4ab^3 - 4a^2bc$$

$$= 4a^4 + 4ac^3 + 4ab^3 - 12a^2bc$$

$$= 4a(a^3 + b^3 + c^3 - 3abc)$$

If roots of given equation are equal, then

$$\text{Disc.} = 0$$

$$4a(a^3 + b^3 + c^3 - 3abc) = 0$$

$$4a = 0 \quad \text{or} \quad a^3 + b^3 + c^3 - 3abc = 0$$

As $4 \neq 0$, or $\boxed{a^3 + b^3 + c^3 = 3abc}$

So, $a = 0$

Q.8 Show that the roots of the following equations are rational

Solution:

(i) $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

$$Ax^2 + Bx + C = 0$$

$$A = a(b-c), B = b(c-a), C = c(a-b)$$

$$\text{Disc.} = B^2 - 4AC$$

$$= [b(c-a)]^2 - 4[a(b-c)c(a-b)]$$

$$= b^2(c-a)^2 - 4[ac(b-c)(a-b)]$$

$$= b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ca + bc)$$

$$= b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4c^2a^2 - 4abc^2$$

$$= a^2b^2 + b^2c^2 + 4c^2a^2 + 2ab^2c - 4abc^2 - 4a^2bc$$

$$= (ab)^2 + (bc)^2 + (-2ca)^2 + 2(ab)(bc)$$

$$+ 2(bc)(-2ca) + 2(-2ca)(ab)$$

By using formula

$$(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2BC + 2CA$$

So, $= [(ab) + (bc) + (-2ca)]^2$

$$= (ab + bc - 2ca)^2 > 0$$

As Disc. is perfect square so the roots of equations are rational.

(ii) $(a+2b)x^2 + 2(a+b+c)x + (a+2c) = 0$

Solution:

$$A = a+2b, B = 2(a+b+c), C = (a+2c)$$

$$\text{Disc.} = B^2 - 4AC$$

$$= [2(a+b+c)]^2 - 4(a+2b)(a+2c)$$

$$= 4[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca] - 4(a^2 + 2ac + 2ab + 4bc)$$

$$= \cancel{4a^2} + 4b^2 + 4c^2 + \cancel{8ab} + \cancel{8bc} + \cancel{8ca} - \cancel{4a^2} - \cancel{8ac} - \cancel{8ab} - 16bc$$

$$= 4b^2 + 4c^2 + 8bc - 16bc$$

$$= 4b^2 + 4c^2 - 8bc$$

$$= (2b)^2 + (2c)^2 - 2(2b)(2c)$$

$$= (2b-2c)^2 > 0$$

As Disc. is perfect square therefore roots, are rational (real and unequal)

Q.9. For all values of K, Prove that the Roots, of the Equation.

$$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0, k \neq 0 \text{ are real}$$

Solution. $a=1, b = -2\left(k + \frac{1}{k}\right), c = 3$

$$\text{Disc.} = b^2 - 4ac$$

$$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(3)$$

$$= 4\left(k + \frac{1}{k}\right)^2 - 12$$

$$= 4\left[\left(k + \frac{1}{k}\right)^2 - 3\right]$$

$$= 4\left[k^2 + \frac{1}{k^2} + 2(k)\left(\frac{1}{k}\right) - 3\right]$$

$$= 4\left[k^2 + \frac{1}{k^2} + 2 - 3\right]$$

$$= 4\left[k^2 + \frac{1}{k^2} - 1\right]$$

$$= 4\left[(k)^2 + \left(\frac{1}{k}\right)^2 - 2 + 1\right]$$

$$= 4\left[(k)^2 + \left(\frac{1}{k}\right)^2 - 2(k)\left(\frac{1}{k}\right) + 1\right]$$

$$= 4\left[\left(k - \frac{1}{k}\right)^2 + 1\right] > 0$$

As Disc. is positive so roots of the given equation are real.

Q.10. Show that the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are real.

Solution: $(b - c)x^2 + (c - a)x + (a - b) = 0$

$$Ax^2 + Bx + C = 0$$

$$A = (b - c), B = (c - a), C = (a - b)$$

$$\text{Disc} = B^2 - 4AC$$

$$= (c - a)^2 - 4(b - c)(a - b)$$

$$= (c^2 + a^2 - 2ca) - 4(ab - b^2 - ca + bc)$$

$$= c^2 + a^2 - 2ca - 4ab + 4b^2 + 4ca - 4bc$$

$$= c^2 + a^2 + 4b^2 + 2ca - 4bc - 4ab$$

$$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ca$$

$$= (a)^2 + (-2b)^2 + (c)^2 + 2(a)(-2b) + 2(-2b)(c) + (c)(a)$$

$$= (a - 2b + c)^2$$

Perfect square shows that the roots of the given equation are real.

Cube Roots of Unity:

Solution:

Let a number x be the cube root of unity

$$\text{i.e., } x = (1)^{1/3}$$

$$\text{or } x^3 = 1$$

$$\Rightarrow x^3 - 1 = 0$$

$$(x^3) - (1)^3 = 0$$

$$(x - 1)(x^2 + x + 1) = 0$$

$$[\text{Using } a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$$

$$\text{Either } x - 1 = 0 \text{ or } x^2 + x + 1 = 0$$

$$\Rightarrow x = 1$$

Now we solve $x^2 + x + 1 = 0$ by formula

$$a = 1, b = 1, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ Three cube roots of unity are

$$1, \frac{-1 + i\sqrt{3}}{2} \text{ and } \frac{-1 - i\sqrt{3}}{2}, \text{ where } i = \sqrt{-1}$$

Recognise complex cube roots of unity as ω and ω^2

The two complex cube roots of unity are

$$\frac{-1 + \sqrt{-3}}{2} \text{ and } \frac{-1 - \sqrt{-3}}{2}$$

If we name any one of these as ω (pronounced as omega), then the other is ω^2 .

Properties of cube root of unity

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof: The complex cube roots of unity are

$$\frac{-1 + \sqrt{-3}}{2} \text{ and } \frac{-1 - \sqrt{-3}}{2}$$

We prove that

$$\left(\frac{-1 + \sqrt{-3}}{2} \right)^2 = \frac{-1 - \sqrt{-3}}{2} \text{ and } \left(\frac{-1 - \sqrt{-3}}{2} \right)^2 = \frac{-1 + \sqrt{-3}}{2}$$

$\left(\frac{-1 + \sqrt{-3}}{2} \right)^2$ $= \frac{(-1)^2 + (\sqrt{-3})^2 + 2(-1)\sqrt{-3}}{(2)^2}$ $= \frac{1 + (-3) - 2\sqrt{-3}}{4}$ $= \frac{-2 - 2\sqrt{-3}}{4}$ $= \frac{2(-1 - \sqrt{-3})}{4}$ $= \frac{-1 - \sqrt{-3}}{2}$	$\left(\frac{-1 - \sqrt{-3}}{2} \right)^2$ $= \frac{(-1)^2 + (\sqrt{-3})^2 - 2(-1)\sqrt{-3}}{(2)^2}$ $= \frac{1 + (-3) + 2\sqrt{-3}}{4}$ $= \frac{-2 + 2\sqrt{-3}}{4}$ $= \frac{2(-1 + \sqrt{-3})}{4}$ $= \frac{-1 + \sqrt{-3}}{2}$
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Thus, each of the complex cube root of unity is the square of the other, that is,

$$\text{If } \omega = \frac{-1 + \sqrt{-3}}{2}, \text{ then } \omega^2 = \frac{-1 - \sqrt{-3}}{2} \text{ and if}$$

$$\omega = \frac{-1 - \sqrt{-3}}{2}, \omega^2 = \frac{-1 + \sqrt{-3}}{2}$$

(b) Prove that the product of three cube roots of unity is one.

Proof: Three cube roots of unity are

$$1, \omega = \frac{-1 + \sqrt{-3}}{2} \text{ and } \omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

The product of cube roots of unity

$$1 \cdot \omega \cdot \omega^2 = (1) \left(\frac{-1 + \sqrt{-3}}{2} \right) \left(\frac{-1 - \sqrt{-3}}{2} \right)$$

$$1 \cdot \omega \cdot \omega^2 = \frac{(-1)^2 - (\sqrt{-3})^2}{4}$$

$$1 \cdot \omega \cdot \omega^2 = \frac{1 - (-3)}{4} = \frac{1 + 3}{4} = \frac{4}{4} = 1$$

$$\text{i.e., } (1)(\omega)(\omega^2) = 1 \text{ or } \omega^3 = 1$$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof: We know that

$$\omega^3 = 1 \Rightarrow \omega \cdot \omega^2 = 1, \text{ so}$$

$$\omega = \frac{1}{\omega^2} \text{ or } \omega^2 = \frac{1}{\omega}$$

Thus, each complex cube root of unity is reciprocal of the other.

(d) Prove that the sum of all the cube roots of unity is zero.

$$\text{i.e., } 1 + \omega + \omega^2 = 0$$

Proof: The cube roots of unity are

$$1, \frac{-1 + \sqrt{-3}}{2} \text{ and } \frac{-1 - \sqrt{-3}}{2}$$

$$\text{If } \omega = \frac{-1 + \sqrt{-3}}{2}, \text{ and } \omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

The sum of all the roots

$$1 + \omega + \omega^2 = 1 + \frac{-1 + \sqrt{-3}}{2} + \frac{-1 - \sqrt{-3}}{2}$$

$$= \frac{2 - 1 + \cancel{\sqrt{-3}} - 1 - \cancel{\sqrt{-3}}}{2}$$

$$= \frac{2 - 2}{2} = \frac{0}{2} = 0$$

$$\text{Thus, } 1 + \omega + \omega^2 = 0$$

Use of properties of cube roots of unity to solve appropriate problems.

We can reduce the higher powers of ω into 1, ω and ω^2 .

$$\begin{aligned} \text{e.g., } \omega^7 &= (\omega^3)^2 \cdot \omega = (1)^2 \cdot \omega = \omega \\ \omega^{23} &= (\omega^3)^7 \cdot \omega^2 = (1)^7 \cdot \omega^2 = \omega^2 \\ \omega^{63} &= (\omega^3)^{21} \cdot (1)^{21} = 1 \end{aligned}$$

$$\omega^{-5} = \frac{1}{\omega^5} = \frac{1}{\omega^3 \cdot \omega^2} = \frac{1}{1 \cdot \omega^2} = \frac{\omega^3}{\omega^2} = \omega$$

$$(\because \omega^3 = 1)$$

$$\omega^{-16} = \frac{1}{\omega^{16}} = \frac{1}{\omega^{15} \cdot \omega} = \frac{1}{(\omega^3)^5 \cdot \omega}$$

$$= \frac{1}{(1)^5 \cdot \omega} = \frac{\omega^3}{\omega} = \omega^2$$

$$\omega^{-27} = \frac{1}{\omega^{27}} = \frac{1}{(\omega^3)^9} = \frac{1}{(1)^9} = 1$$

Example 1: Evaluate $(-1 + \sqrt{-3})^8 + (-1 - \sqrt{-3})^8$.

Solution: $(-1 + \sqrt{-3})^8 + (-1 - \sqrt{-3})^8$

$$\begin{aligned} &= \left[2 \left(\frac{-1 + \sqrt{-3}}{2} \right) \right]^8 + \left[2 \left(\frac{-1 - \sqrt{-3}}{2} \right) \right]^8 \\ &= (2\omega)^8 + (2\omega^2)^8 \\ &= 256 \omega^8 + 256 \omega^{16} \\ &= 256 [\omega^8 + \omega^{16}] \\ &= 256 [(\omega^3)^2 \cdot \omega^2 + (\omega^3)^5 \cdot \omega] (\because \omega^3 = 1) \\ &= 256 [\omega^2 + \omega] \quad (\omega + \omega^2 = -1) \\ &= 256 (-1) = -256 \end{aligned}$$

Example 2: Prove that

$$x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y).$$

Solution: $x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y).$

$$\begin{aligned} \text{R.H.S} &= (x - y)(x - \omega y)(x - \omega^2 y) \\ &= (x - y)[x^2 - \omega^2 xy - \omega xy + \omega^3 y^2] \\ &= (x - y)[x^2 - xy(\omega^2 + \omega) + (1)y^2] \\ &= (x - y)[x^2 - xy(-1) + y^2] \\ &= (x - y)[x^2 + xy + y^2] \\ &= x^3 - y^3 = \text{L.H.S} \end{aligned}$$