

EXERCISE 2.3

Q.1 Without solving, Find the sum and product of the Roots of following Quadratic Equations.

(i) $x^2 - 5x + 3 = 0$

Solution: $x^2 - 5x + 3 = 0$

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = 1, b = -5, c = 3$$

$$\text{Sum of roots} = S = \frac{-b}{a} = -\left(\frac{-5}{1}\right) = 5$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{3}{1} = 3$$

(ii) $3x^2 + 7x - 11 = 0$

Solution: $3x^2 + 7x - 11 = 0$

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = 3, b = 7, c = -11$$

$$\text{Sum of roots} = S = \frac{-b}{a} = \frac{-7}{3}$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{-11}{3}$$

(iii) $px^2 - qx + r = 0$

Solution: $px^2 - qx + r = 0$

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = p, b = -q, c = r$$

$$\text{Sum of roots} = S = \frac{-b}{a} = -\left(\frac{-q}{p}\right) = \frac{q}{p}$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{r}{p}$$

(iv) $(a+b)x^2 - ax + b = 0$

Solution: $(a+b)x^2 - ax + b = 0$

$$Ax^2 + Bx + C = 0$$

$$\Rightarrow A = a+b, B = -a, C = b$$

$$\text{Sum of roots} = S = \frac{-B}{A} = -\left(\frac{-a}{a+b}\right) = \frac{a}{a+b}$$

$$\text{Product of roots} = P = \frac{C}{A} = \frac{b}{a+b}$$

(v) $(\ell + m)x^2 + (m + n)x + n - \ell = 0$

Solution: $(\ell + m)x^2 + (m + n)x + n - \ell = 0$

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = \ell + m, b = m + n, c = n - \ell$$

$$\text{Sum of roots} = S = \frac{-b}{a} = \frac{-(m+n)}{\ell+m}$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{n-\ell}{\ell+m}$$

(vi) $7x^2 - 5mx + 9n = 0$

Solution: $7x^2 - 5mx + 9n = 0$

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = 7, b = -5m, c = 9n$$

$$\text{Sum of roots} = \frac{-b}{a} = -\left(\frac{-5m}{7}\right) = \frac{5m}{7}$$

$$\text{Product of roots} = \frac{c}{a} = \frac{9n}{7}$$

Q. 2 Find the Value of k if.

(i) Sum of the roots of the equation $2kx^2 - 3x + 4k = 0$ is twice the product of the roots.

Solution:

$$2kx^2 - 3x + 4k = 0$$

$$ax^2 + bx + c = 0$$

$$a = 2k, b = -3, c = 4k$$

Let α, β be the Roots of Equation

$$\text{Sum of the roots} = \alpha + \beta = \frac{-b}{a}$$

$$S = \alpha + \beta = \frac{-(-3)}{2k} = \frac{3}{2k}$$

$$\text{product of the Roots} = \alpha\beta = \frac{c}{a}$$

$$P = \alpha\beta = \frac{4k}{2k} = 2$$

Given condition

$$S = 2P$$

$$\frac{3}{2k} = 2(2)$$

$$\frac{3}{2k} = 4$$

$$3 = 4(2k)$$

$$3 = 8k$$

$$\frac{3}{8} = k \Rightarrow \boxed{k = \frac{3}{8}}$$

(ii). Sum of the roots of the equation

$$x^2 + (3k - 7)x + 5k = 0 \text{ is } \frac{3}{2} \text{ times the}$$

product of roots.

Solution:

$$1x^2 + (3k - 7)x + 5k = 0$$

$$ax^2 + bx + c = 0$$

$$a = 1, b = 3k - 7, c = 5k$$

Let α, β be the roots of the given equation.

Sum of the roots

$$S = \alpha + \beta = \frac{-b}{a}$$

$$S = \frac{-(3K - 7)}{1}$$

$$S = -3K + 7$$

$$\text{Product of the roots} = P = \alpha \beta = \frac{c}{a}$$

$$P = \frac{5k}{1}$$

$$P = 5k$$

Given condition

$$S = \frac{3}{2} P$$

$$-3k + 7 = \frac{3}{2} (5k)$$

$$2(-3k + 7) = 3(5k)$$

$$-6k + 14 = 15k$$

$$14 = 15k + 6k$$

$$14 = 21k$$

$$k = \frac{14}{21} = \frac{2}{3}$$

$$\boxed{k = \frac{2}{3}}$$

Q. 3 Find k if,

(i) Sum of the squares of the roots of the equation. $4kx^2 + 3kx - 8 = 0$ is 2

Solution:

$$4kx^2 + 3kx - 8 = 0 \text{ is } 2$$

$$ax^2 + bx + c = 0$$

$$a = 4k, b = 3k, c = -8$$

Sum of roots

$$S = \alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-3k}{4k}$$

$$\alpha + \beta = \frac{-3}{4}$$

$$\text{Product of roots} = P = \alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{-8}{4k}$$

$$\alpha\beta = \frac{-2}{k}$$

Given that sum of square of roots is 2 i.e.

$$\alpha^2 + \beta^2 = 2$$

$$\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = 2$$

$$(\alpha + \beta)^2 - 2\alpha\beta = 2$$

$$\text{Put } \alpha + \beta = \frac{-3}{4} \text{ and } \alpha\beta = \frac{-2}{k}$$

$$\left(\frac{-3}{4}\right)^2 - 2\left(\frac{-2}{k}\right) = 2$$

$$\frac{9}{16} + \frac{4}{k} = 2$$

$$\frac{9k + 64}{16k} = 2$$

$$9k + 64 = 2(16k)$$

$$9k + 64 = 32k$$

$$64 = 32k - 9k$$

$$\Rightarrow 23k = 64$$

$$k = \frac{64}{23}$$

$$\text{Required value of } k = \frac{64}{23}$$

(ii) Sum of square of the roots of the equation $x^2 - 2kx + (2k + 1) = 0$ is 6.

Solution: $1x^2 - 2kx + (2k + 1) = 0$

$$ax^2 + bx + c = 0$$

$$a = 1, b = -2k, c = 2k + 1$$

α, β be the Roots of given equation.

$$S = \alpha + \beta = -\frac{b}{a}$$

$$\alpha + \beta = \frac{-(-2k)}{1} = 2k$$

$$\alpha\beta = \frac{c}{a}$$

$$P = \alpha\beta = \frac{2k + 1}{1} = 2k + 1$$

Given condition

$$\alpha^2 + \beta^2 = 6$$

$$\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = 6$$

$$(\alpha + \beta)^2 - 2(\alpha\beta) = 6$$

$$(-2k)^2 - 2(2k + 1) = 6$$

$$4k^2 - 4k - 2 = 6$$

$$4k^2 - 4k - 2 - 6 = 0$$

$$4k^2 - 4k - 8 = 0$$

$$4(k^2 - k - 2) = 0$$

$$\therefore k^2 - k - 2 = 0 \quad (\because 4 \neq 0)$$

$$k^2 - 2k + k - 2 = 0$$

$$k(k - 2) + 1(k - 2) = 0$$

$$(k - 2)(k + 1) = 0$$

$$k - 2 = 0 \text{ or } k + 1 = 0$$

$$k = 2 \text{ or } k = -1$$

$$k = -1, 2$$

Q.4 Find p if

(i) The roots of the equation

$$x^2 - x + p^2 = 0 \text{ differ by unity. } 02(045)$$

Solution: $x^2 - x + p^2 = 0$

$$ax^2 + bx + c = 0$$

$$a = 1, b = -1, c = p^2$$

Let the roots are ' α and $\alpha - 1$ '

$$\text{Sum of roots} = S = \alpha + \alpha - 1 = \frac{-b}{a} = -\left(\frac{-1}{1}\right) = 1$$

$$2\alpha - 1 = 1$$

$$\Rightarrow 2\alpha = 1 + 1$$

$$\Rightarrow 2\alpha = 2 \Rightarrow \boxed{\alpha = 1}$$

$$\text{Product of roots} = P = \alpha(\alpha - 1) = \frac{c}{a} = \frac{p^2}{1} = p^2$$

$$\alpha(\alpha - 1) = p^2$$

Putting the value of $\alpha = 1$

$$1(1 - 1) = p^2$$

$$\Rightarrow p^2 = 0$$

$$\Rightarrow \boxed{p = 0}$$

(ii) Find p if the roots of the equation

$$x^2 + 3x + p - 2 = 0 \text{ differ by 2.}$$

Solution: $x^2 + 3x + p - 2 = 0$

$$a = 1, b = 3, c = p - 2$$

Let ' α ' and $\alpha - 2$ are the roots

Sum of roots

$$S = \alpha + \alpha - 2 = \frac{-b}{a} = \frac{-3}{1} = -3$$

$$2\alpha - 2 = -3$$

$$2\alpha = -3 + 2$$

$$\Rightarrow 2\alpha = -1 \Rightarrow \boxed{\alpha = \frac{-1}{2}} \dots (i)$$

Product of roots

$$P = \alpha(\alpha - 2) = \frac{c}{a} = \frac{p - 2}{1} = p - 2$$

$$\alpha(\alpha - 2) = p - 2$$

Putting the value of α from equation (i)

$$-\frac{1}{2} \left(\frac{-1}{2} - 2 \right) = p - 2$$

$$-\frac{1}{2} \left(\frac{-5}{2} \right) = p - 2$$

$$\frac{5}{4} = p - 2$$

$$\Rightarrow p = \frac{5}{4} + 2$$

$$p = \frac{5 + 8}{4} \Rightarrow \boxed{p = \frac{13}{4}}$$

Q.5 Find m if

- (i) The roots of the equation $x^2 - 7x + 3m - 5 = 0$
Satisfy the relation $3\alpha + 2\beta = 4$

Solution:

Let α, β be the roots of given equation.

$$1x^2 - 7x + 3m - 5 = 0$$

$$a = 1, b = -7, c = 3m - 5$$

Sum of roots

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-7)}{1} = 7 \dots\dots\dots (i)$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{3m-5}{1} = 3m-5 \dots\dots\dots (ii)$$

Since $3\alpha + 2\beta = 4$ (given).....(iii)

From equation (i)

$$\alpha + \beta = 7$$

$$\beta = 7 - \alpha$$

Put $\beta = 7 - \alpha$ in equation (iii)

$$3\alpha + 2(7 - \alpha) = 4$$

$$3\alpha + 14 - 2\alpha = 4$$

$$\alpha + 14 = 4$$

$$\alpha = 4 - 14$$

$$\boxed{\alpha = -10}$$

Put $\alpha = -10$ in $\beta = 7 - \alpha$

$$\beta = 7 - (-10)$$

$$\beta = 7 + 10$$

$$\boxed{\beta = 17}$$

Put $\alpha = -10$ and $\beta = 17$ in equation (ii)

$$\alpha\beta = 3m - 5$$

$$(-10)(17) = 3m - 5$$

$$-170 = 3m - 5$$

$$-170 + 5 = 3m$$

$$-165 = 3m$$

$$\frac{-165}{3} = m$$

$$-55 = m \Rightarrow \boxed{m = -55}$$

~~(ii)~~ Find m if the roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$

Solution:

Let α, β be the roots of the equation.

$$1x^2 + 7x + 3m - 5 = 0$$

$$a = 1, b = 7, c = 3m - 5$$

Sum of roots

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-7}{1}$$

$$\alpha + \beta = -7 \dots\dots\dots (i)$$

Product of roots

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{3m-5}{1}$$

$$\alpha\beta = 3m - 5 \dots\dots\dots (ii)$$

$$3\alpha - 2\beta = 4 \dots\dots\dots (iii) \quad \text{(Given)}$$

From equation (i)

$$\alpha + \beta = -7$$

$$\beta = -7 - \alpha$$

Putting the value of β in equation (iii)

$$3\alpha - 2\beta = 4$$

$$3\alpha - 2(-7 - \alpha) = 4$$

$$3\alpha + 14 + 2\alpha = 4$$

$$5\alpha = 4 - 14$$

$$5\alpha = -10$$

$$\alpha = \frac{-10}{5}$$

$$\boxed{\alpha = -2}$$

Put $\alpha = -2$ in equation (i)

$$\alpha + \beta = -7$$

$$-2 + \beta = -7$$

$$\beta = -7 + 2$$

$$\boxed{\beta = -5}$$

Now Put $\alpha = -2$ and $\beta = -5$ in equation (ii)

$$\alpha\beta = 3m - 5$$

$$(-2)(-5) = 3m - 5$$

$$10 = 3m - 5$$

$$10 + 5 = 3m$$

$$15 = 3m$$

$$m = \frac{15}{3}$$

$\Rightarrow$

$$\boxed{m = 5}$$

- (iii) Find m if the roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$

Solution: Let α, β be the roots of the equation

$$3x^2 - 2x + 7m + 2 = 0$$

$$ax^2 + bx + c = 0$$

$$a = 3, b = -2, c = 7m + 2$$

Sum of roots

$$\alpha + \beta = -\frac{b}{a} = \frac{-(-2)}{3} = \frac{2}{3} \dots\dots\dots (i)$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{7m+2}{3} \dots\dots\dots (ii)$$

$$7\alpha - 3\beta = 18 \dots\dots\dots (iii)$$

From equation (i)

$$\alpha + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - \alpha$$

Putting this value in eq. (iii)

$$7\alpha - 3\beta = 18$$

$$7\alpha - 3\left(\frac{2}{3} - \alpha\right) = 18$$

$$7\alpha - \frac{6}{\cancel{3}} + 3\alpha = 18$$

$$7\alpha - 2 + 3\alpha = 18$$

$$10\alpha = 18 + 2$$

$$10\alpha = 20$$

$$\alpha = \frac{20}{10}$$

$$\boxed{\alpha = 2}$$

Putting $\alpha = 2$ in equation (i)

$$\alpha + \beta = \frac{2}{3}$$

$$2 + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - 2$$

$$\beta = \frac{2-6}{3} = \frac{-4}{3}$$

$$\boxed{\beta = \frac{-4}{3}}$$

Now putting the value of α and β in equation (ii)

$$\alpha\beta = \frac{7m+2}{3}$$

$$2\left(-\frac{4}{3}\right) = \frac{7m+2}{3}$$

$$-\frac{8}{\cancel{3}} \times \cancel{3} = 7m+2$$

$$\Rightarrow 7m+2 = -8$$

$$7m = -8 - 2$$

$$\Rightarrow m = \frac{-10}{7}$$

Q.6 Find m , if sum and product of the roots of the following equations is equal to given number λ .

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Solution:

$$(2m+3)x^2 + (7m-5)x + (3m-10) = 0$$

$$ax^2 + bx + c = 0$$

$$a = 2m+3, b = 7m-5, c = 3m-10$$

Let α, β are the roots of the given equation, then

Sum of Roots

$$\alpha + \beta = -\frac{b}{a} = -\frac{(7m-5)}{2m+3} = \frac{5-7m}{2m+3}$$

Product of Roots

$$\alpha\beta = \frac{c}{a} = \frac{3m-10}{2m+3}$$

As given that

$$\alpha + \beta = \lambda \dots\dots\dots (i), \alpha\beta = \lambda \dots\dots\dots (ii)$$

From (i) and (ii)

$$\alpha + \beta = \alpha\beta$$

$$\frac{5-7m}{2m+3} = \frac{3m-10}{2m+3}$$

$$5-7m = 3m-10$$

$$5+10 = 3m+7m$$

$$15 = 10m$$

$$\frac{15}{10} = m$$

$$\boxed{m = \frac{3}{2}}$$

$$(ii) \quad 4x^2 - (3 + 5m)x - (9m - 17) = 0$$

Solution:

Let α, β be the roots of the equation.

$$4x^2 - (3 + 5m)x - (9m - 17) = 0$$

$$\therefore ax^2 + bx + c = 0$$

$$\therefore a = 4, b = -(3 + 5m), c = -(9m - 17)$$

Sum of roots

$$\alpha + \beta = \frac{-b}{a} = -\left(-\frac{(3 + 5m)}{4}\right) = \frac{3 + 5m}{4}$$

Product of roots

$$\alpha\beta = \frac{c}{a} = -\left(\frac{9m - 17}{4}\right)$$

Let $\alpha + \beta = \lambda \dots (i)$ and $\alpha\beta = \lambda \dots (ii)$

from (i) and (ii)

$$\alpha + \beta = \alpha\beta$$

$$\frac{3 + 5m}{4} = \frac{-(9m - 17)}{4}$$

$$3 + 5m = -9m + 17$$

$$9m + 5m = 17 - 3$$

$$14m = 14$$

$$\Rightarrow m = \frac{14}{14} \Rightarrow \boxed{m = 1}$$

Symmetric functions of the roots of a quadratic equation.

Definition:

Symmetric Functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.

For example, if

$$f(\alpha, \beta) = \alpha^2 + \beta^2, \text{ then}$$

$$f(\beta, \alpha) = \beta^2 + \alpha^2 = \alpha^2 + \beta^2$$

$$(\because \beta^2 + \alpha^2 = \alpha^2 + \beta^2)$$

$$= f(\alpha, \beta)$$

Example: Find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$, if $\alpha = 2, \beta = 1$. Also find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$ if $\alpha = 1, \beta = 2$.

Solution:

(i) When $\alpha = 2$ and $\beta = 1$,

$$\begin{aligned} \alpha^3 + \beta^3 + 3\alpha\beta &= (2)^3 + (1)^3 + 3(2)(1) \\ &= 8 + 1 + 6 = 15 \end{aligned}$$

(ii) When $\alpha = 1$ and $\beta = 2$,

$$\begin{aligned} \alpha^3 + \beta^3 + 3\alpha\beta &= (1)^3 + (2)^3 + 3(1)(2) \\ &= 1 + 8 + 6 = 15 \end{aligned}$$

The expression $\alpha^3 + \beta^3 + 3\alpha\beta$ represents a symmetric function of α and β .

Evaluate a symmetric function of roots of a quadratic equation in terms of its co-efficients.

If α, β are the roots of the quadratic equation

$$ax^2 + bx + c = 0, (a \neq 0) \dots (i)$$

$$\text{Then } \alpha + \beta = \frac{-b}{a} \dots (ii)$$

$$\text{and } \alpha\beta = \frac{c}{a} \dots (iii)$$

The functions given in equations (ii) and (iii) are the symmetric functions for the quadratic equation (i).

Some more symmetric functions of two variables α, β are given below:

$$(i) \quad \alpha^2 + \beta^2 \quad (ii) \quad \alpha^3 + \beta^3$$

$$(iii) \quad \frac{1}{\alpha} + \frac{1}{\beta} \quad (iv) \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

Example 1: If α, β are the roots of the quadratic equation $px^2 + qx + r = 0$, ($p \neq 0$) then evaluate $\alpha^2\beta + \alpha\beta^2$

Solution:

Since α, β are the roots of $px^2 + qx + r = 0$, therefore,

$$\alpha + \beta = \frac{-b}{a} = -\frac{q}{p} \quad \text{and} \quad \alpha\beta = \frac{r}{p}$$

$$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$$

$$= \frac{r}{p} \left(-\frac{q}{p} \right) = \frac{-qr}{p^2}$$

Example 2: If α, β are the roots of the equation

$2x^2 + 3x + 4 = 0$, then find the value of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

Since, α, β are the roots of the equation

$2x^2 + 3x + 4 = 0$, therefore,

$$\alpha + \beta = \frac{-b}{a}$$

$$= -\frac{3}{2}$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{4}{2} = 2$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{-3}{2}\right)^2 - 2(2)$$

$$= \frac{9}{4} - 4$$

$$= \frac{9-16}{4}$$

$$= \frac{-7}{4}$$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$$= (\alpha + \beta) \frac{1}{\alpha\beta}$$

$$= \left(-\frac{3}{2}\right) \left(\frac{1}{2}\right) = \frac{-3}{4}$$