

EXERCISE 2.4

Q.1 If α, β are the roots of the equations $x^2 + px + q = 0$ then evaluate

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3\beta + \alpha\beta^3$ (iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

Solution:

$$a = 1, b = p, c = q$$

Sum of roots,

$$\alpha + \beta = \frac{-b}{a} = \frac{-p}{1} = -p$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q$$

(i) $\alpha^2 + \beta^2$

As

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

or $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (-p)^2 - 2(q)$$

$$= p^2 - 2q$$

(ii) $\alpha^3\beta + \alpha\beta^3$

$$= \alpha\beta(\alpha^2 + \beta^2)$$

$$= \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= q[(-p)^2 - 2q]$$

$$= q(p^2 - 2q)$$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{1}{\alpha\beta}(\alpha^2 + \beta^2)$$

$$= \frac{1}{\alpha\beta}[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= \frac{1}{q}[(-p)^2 - 2 \times q]$$

$$= \frac{1}{q}(p^2 - 2q)$$

Q.2 If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $\alpha^2\beta^2$ (iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$ (iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

Solution: $4x^2 - 5x + 6 = 0$
 $ax^2 + bx + c = 0$
 $a = 4, b = -5, c = 6$

Sum of roots,

$$\alpha + \beta = \frac{-b}{a} = -\frac{(-5)}{4} = \frac{5}{4}$$

Product of roots,

$$\alpha\beta = \frac{c}{a} = \frac{6}{4}$$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{5}{4}}{\frac{6}{4}} = \frac{5}{6}$

(ii) $\alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{6}{4}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta + \alpha}{\alpha^2\beta^2} = \frac{\alpha + \beta}{(\alpha\beta)^2}$
 $= \frac{\frac{5}{4}}{\left(\frac{6}{4}\right)^2} = \frac{5}{36} = \frac{5}{4} \times \frac{16}{36} = \frac{5}{9}$

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$

Using Formula: $(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3(\alpha\beta)(\alpha + \beta)$

$$\Rightarrow \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3(\alpha\beta)(\alpha + \beta)$$

$$= \frac{(\alpha + \beta)^3 - 3(\alpha\beta)(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{\left(\frac{5}{4}\right)^3 - 3\left(\frac{6}{4}\right)\left(\frac{5}{4}\right)}{\left(\frac{6}{4}\right)} = \left(\frac{125}{64} - \frac{90}{16}\right) \frac{4}{6}$$

$$= \left(\frac{125 - 360}{64}\right) \frac{4}{6} = \frac{-235}{96}$$

Q.3 If α, β are the roots of the equation $lx^2 + mx + n = 0$ ($l \neq 0$) then find the value of

(i) $\alpha^3\beta^2 + \alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution: $lx^2 + mx + n = 0$

$$ax^2 + bx + c = 0$$

$$a = l, b = m, c = n$$

If α, β be the Roots of given equation

$$\text{Sum of roots} = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-m}{l}$$

$$\text{Product of Roots} = \frac{c}{a}$$

$$\alpha\beta = \frac{n}{l}$$

(i) $\alpha^3\beta^2 + \alpha^2\beta^3$
 $= \alpha^2\beta^2(\alpha + \beta)$
 $= (\alpha\beta)^2(\alpha + \beta)$
 $= \left(\frac{n}{l}\right)^2 \left(\frac{-m}{l}\right)$
 $= \left(\frac{n^2}{l^2}\right) \left(\frac{-m}{l}\right)$
 $= \frac{-mn^2}{l^3}$

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
 $= \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$
 $= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$
 $= \frac{\left(\frac{-m}{l}\right)^2 - 2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2} = \frac{\frac{m^2}{l^2} - \frac{2n}{l}}{\frac{n^2}{l^2}}$
 $= \left(\frac{m^2 - 2nl}{n^2}\right) \cdot \frac{l^2}{l^2}$
 $= \left(\frac{m^2 - 2nl}{n^2}\right)$
 $= \frac{1}{n^2}(m^2 - 2nl)$

Formation of a quadratic equation.

If α and β are the roots of the required quadratic equation.

$$\begin{aligned}\text{Let } x &= \alpha \quad \text{and} \quad x = \beta \\ \text{i.e., } x - \alpha &= 0, \quad x - \beta = 0 \\ \text{and } (x - \alpha)(x - \beta) &= 0 \\ x^2 - \beta x - \alpha x + \alpha\beta &= 0 \\ x^2 - \alpha x - \alpha x + \alpha\beta &= 0 \\ x^2 - (\alpha + \beta)x + \alpha\beta &= 0\end{aligned}$$

which is the required quadratic equation in the standard form.

Find a quadratic equation from given roots and establish the formula

$$x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0.$$

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0, \quad (a \neq 0) \dots (i)$$

$$\text{Then } \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\text{Rewrite equation (i) as } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\text{or } x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0, \\ \text{that is, } x^2 - Sx + P = 0,$$

$$\text{where } S = \alpha + \beta \text{ and } P = \alpha\beta$$

Example 1: Form a quadratic equation with roots 3 and 4.

Solution: Since 3 and 4 are the roots of the required quadratic equation, therefore,

$$S = \text{Sum of the roots} = 3 + 4 = 7$$

$$P = \text{Product of the roots} = (3)(4) = 12$$

As $x^2 - Sx + P = 0$, so the required quadratic equation is $x^2 - 7x + 12 = 0$

Form quadratic equations whose roots are of the type.

$$(i) \quad 2\alpha + 1, 2\beta + 1 \quad (ii) \quad \alpha^2, \beta^2$$

$$(iii) \quad \frac{1}{\alpha}, \frac{1}{\beta} \quad (iv) \quad \frac{\alpha}{\beta}, \frac{\beta}{\alpha}$$

$$(v) \quad \alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta} \text{ where } \alpha, \beta \text{ are the roots} \\ \text{of a given quadratic equation.}$$

Example 2: If α, β are the roots of the equation $2x^2 - 3x - 5 = 0$, form quadratic equations having roots.

$$(i) \quad 2\alpha + 1, 2\beta + 1 \quad (ii) \quad \alpha^2, \beta^2$$

$$(iii) \quad \frac{1}{\alpha}, \frac{1}{\beta} \quad (iv) \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$(v) \quad \alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$$

Solution: As α, β are the roots of the equation

$$2x^2 - 3x - 5 = 0,$$

$$ax^2 + bx + c = 0$$

$$a = 2, b = -3, c = -5$$

$$\text{Therefore, } \alpha + \beta = -\frac{b}{a} = -\left(\frac{-3}{2}\right) = \frac{3}{2}$$

$$\text{and } \alpha\beta = \frac{c}{a} = \frac{-5}{2} = -\frac{5}{2}$$

$$(i) \quad 2\alpha + 1, 2\beta + 1$$

Sum of the roots

$$S = 2\alpha + 1 + 2\beta + 1$$

$$S = 2(\alpha + \beta) + 2$$

$$S = 2\left(\frac{3}{2}\right) + 2 = 5 \Rightarrow \boxed{S = 5}$$

Product of the roots

$$P = (2\alpha + 1)(2\beta + 1)$$

$$P = 4\alpha\beta + 2\alpha + 2\beta + 1$$

$$P = 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$P = 4\left(-\frac{5}{2}\right) + 2\left(\frac{3}{2}\right) + 1$$

$$P = -10 + 3 + 1 = -6 \Rightarrow \boxed{P = -6}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - 5x - 6 = 0$$

$$(ii) \quad \alpha^2, \beta^2$$

Sum of the roots

$$S = \alpha^2 + \beta^2$$

$$S = (\alpha + \beta)^2 - 2\alpha\beta$$

$$S = \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)$$

$$S = \frac{9}{4} + 5 = \frac{9+20}{4}$$

$$\boxed{S = \frac{29}{4}}$$

Product $P = \alpha^2 \cdot \beta^2 = (\alpha\beta)^2$

$$P = \left(-\frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow \boxed{P = \frac{25}{4}}$$

Using $x^2 - Sx + p = 0$, we have

$$x^2 - \frac{29}{4}x + \frac{25}{4} = 0 \Rightarrow 4x^2 - 29x + 25 = 0$$

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum $S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = (\alpha + \beta) \cdot \frac{1}{\alpha\beta}$

$$S = \frac{3}{2} \cdot \left(-\frac{2}{5}\right) \quad \left(\because \alpha\beta = -\frac{5}{2}\right)$$

$$\boxed{S = -\frac{3}{5}}$$

Product $P = \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = -\frac{2}{5} \Rightarrow \boxed{P = -\frac{2}{5}}$

Using $x^2 - Sx + p = 0$, we have

$$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{2}{5}\right) = 0$$

$$\Rightarrow 5x^2 + 3x - 2 = 0$$

(iv) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Sum $S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$$S = \left[(\alpha + \beta)^2 - 2\alpha\beta\right] \cdot \frac{1}{\alpha\beta}$$

$$S = \left[\left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)\right] \times \left(-\frac{2}{5}\right)$$

$$S = \left(\frac{9}{4} + 5\right) \times \left(-\frac{2}{5}\right) = \left(\frac{9+20}{4}\right) \times \left(-\frac{2}{5}\right)$$

$$S = \frac{29}{4} \times \left(-\frac{2}{5}\right) = -\frac{29}{10} \Rightarrow \boxed{S = -\frac{29}{10}}$$

Product $P = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(-\frac{29}{10}\right)x + 1 = 0$$

$$\Rightarrow 10x^2 + 29x + 10 = 0$$

(v) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Sum of the roots

$$S = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$$

$$S = \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$$

$$S = (\alpha + \beta) \left(1 + \frac{1}{\alpha\beta}\right)$$

$$S = \frac{3}{2} \left(1 + \frac{2}{5}\right) = \frac{3}{2} \times \left(\frac{5+2}{5}\right) = \frac{3}{2} \times \frac{7}{5} \Rightarrow \boxed{S = \frac{21}{10}}$$

Product of the roots

$$P = (\alpha + \beta) \cdot \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$P = (\alpha + \beta) \left(\frac{\beta + \alpha}{\alpha\beta}\right)$$

$$P = (\alpha + \beta)^2 \times \frac{1}{\alpha\beta} = \left(\frac{3}{2}\right)^2 \times \left(-\frac{2}{5}\right)$$

$$P = \frac{9}{4} \times \left(-\frac{2}{5}\right) = \left(-\frac{9}{10}\right) \Rightarrow \boxed{P = -\frac{9}{10}}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{21}{10}x + \left(-\frac{9}{10}\right) = 0$$

$$\Rightarrow 10x^2 - 21x - 9 = 0$$

Example 3: If α, β are the roots of the equation $x^2 - 7x + 9 = 0$

Form an equation whose roots are 2α and 2β .

Solution:

Since α, β are the roots of the equation $x^2 - 7x + 9 = 0$, therefore,

$$\alpha + \beta = \frac{-b}{a} = -\left(\frac{-7}{1}\right) = 7$$

And $\alpha\beta = \frac{c}{a} = \frac{9}{1} = 9$

The roots of the required equation are $2\alpha, 2\beta$
Sum of roots

$$S = 2\alpha + 2\beta = 2(\alpha + \beta) = 2(7) = 14$$

Product of roots

$$P = (2\alpha)(2\beta) = 4\alpha\beta = 4(9) = 36$$

Thus the required quadratic equation will be

$$x^2 - Sx + P = 0, \text{ that is, } x^2 - 14x + 36 = 0$$