

EXERCISE 2.5

Q.1 Write the quadratic equation having following roots.

- (a). 1, 5 (b) 4, 9 (c) -2, 3
 (d) 0, -3 (e) 2, -6 (f) -1, -7
 (g). $1+i, 1-i$ (h) $3+\sqrt{2}, 3-\sqrt{2}$

(a) 1, 5

Solution: Since 1 and 5 are the roots of the required quadratic equation, therefore

$$\text{Sum of roots} = S = 1 + 5 = 6$$

$$\text{Product of roots} = P = 1 \times 5 = 5$$

As $x^2 - Sx + P = 0$ so the required equation is

$$x^2 - 6x + 5 = 0$$

(b) 4, 9

Solution: Since 4 and 9 are the roots of the required quadratic equation, therefore

$$\text{Sum of roots} = S = 4 + 9 = 13$$

$$\text{Product of roots} = P = 4 \times 9 = 36$$

$$\text{As } x^2 - Sx + P = 0$$

So the required equation is

$$x^2 - 13x + 36 = 0$$

(c) -2, 3

Since -2, 3 are the roots of required quadratic equation, therefore

$$\text{Sum of roots} = S = -2 + 3 = 1$$

$$\text{Product of roots} = P = -2 \times 3 = -6$$

$$\text{As } x^2 - Sx + P = 0$$

Therefore the required quadratic equation is

$$x^2 - x - 6 = 0$$

(d) 0, -3

Since 0, -3 are the roots of required quadratic equation therefore

$$\text{Sum of roots} = S = 0 + (-3) = -3$$

$$\text{Product of roots} = P = 0 \times (-3) = 0$$

$$\text{As } x^2 - Sx + P = 0$$

Therefore the required quadratic equation is

$$x^2 + 3x + 0 = 0 \Rightarrow$$

$$x^2 + 3x = 0$$

(e) 2, -6

Solution: Since 2 and -6 are the roots of the required quadratic equation therefore

$$\text{Sum of roots} = S = 2 + (-6) = 2 - 6 = -4$$

$$\text{Product of roots} = P = 2 \times (-6) = -12$$

As $x^2 - Sx + P = 0$ so the required equation is

$$x^2 + 4x - 12 = 0$$

(f) -1, -7

Solution: Since -1 and -7 are the roots of the required quadratic equation therefore

$$\text{Sum of roots} = S = (-1) + (-7)$$

$$= -1 - 7 = -8$$

$$\text{Product of roots} = P = (-1)(-7) = 7$$

As $x^2 - Sx + P = 0$ so the required equation is

$$x^2 + 8x + 7 = 0$$

(g) $1+i, 1-i$

Solution: Since $1+i$ and $1-i$ are the roots of the required quadratic equation therefore

$$\text{Sum of roots} = S = 1 + 1 = 2$$

$$\text{Product of roots} = P = (1+i)(1-i)$$

$$P = (1)^2 - (i)^2$$

$$P = 1 - (-1)$$

$$P = 1 + 1 = 2$$

As $x^2 - Sx + P = 0$ so the required equation is

$$x^2 - 2x + 2 = 0$$

(h) $3+\sqrt{2}, 3-\sqrt{2}$

Solution: $3+\sqrt{2}$ and $3-\sqrt{2}$ are the roots of the required quadratic equation therefore

$$\text{Sum of roots} = S = 3 + \sqrt{2} + 3 - \sqrt{2} = 6$$

Product of roots

$$P = (3 + \sqrt{2})(3 - \sqrt{2})$$

$$P = (3)^2 - (\sqrt{2})^2$$

$$P = 9 - 2 = 7$$

As $x^2 - Sx + P = 0$, so the required equation is

$$x^2 - 6x + 7 = 0$$

Q.2 If α, β are the roots of the equation $x^2 - 3x + 6 = 0$. Form equation whose roots are (a) $2\alpha + 1, 2\beta + 1$ (b) α^2, β^2

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution: As α, β are the roots of the equation $x^2 - 3x + 6 = 0$

$$a = 1, b = -3, c = 6$$

Therefore,

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-3)}{1} = 3 \Rightarrow \boxed{\alpha + \beta = 3}$$

$$\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6 \Rightarrow \boxed{\alpha\beta = 6}$$

(a) $2\alpha + 1, 2\beta + 1$

Sum of roots

$$S = 2\alpha + 1 + 2\beta + 1$$

$$S = 2\alpha + 2\beta + 2$$

$$S = 2(\alpha + \beta) + 2$$

$$S = 2(3) + 2 = 6 + 2 = 8$$

$$\boxed{S = 8}$$

Product of roots

$$P = (2\alpha + 1)(2\beta + 1)$$

$$P = 4\alpha\beta + 2\alpha + 2\beta + 1$$

$$P = 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$P = 4(6) + 2(3) + 1$$

$$P = 24 + 6 + 1 = 31$$

$$\boxed{P = 31}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - 8x + 31 = 0$$

(b) α^2, β^2

Solution: As α, β are the roots of the equation $x^2 - 3x + 6 = 0$

$$a = 1, b = -3, c = 6$$

Therefore,

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-3)}{1} = 3 \Rightarrow \boxed{\alpha + \beta = 3}$$

$$\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6 \Rightarrow \boxed{\alpha\beta = 6}$$

$$\text{Sum of roots} = S = \alpha^2 + \beta^2$$

$$S = (\alpha + \beta)^2 - 2\alpha\beta$$

$$S = (3)^2 - 2(6)$$

$$S = 9 - 12 = -3$$

$$\boxed{S = -3}$$

$$\text{Product of roots} = P = \alpha^2\beta^2$$

$$P = (\alpha\beta)^2$$

$$P = (6)^2 = 36$$

$$\boxed{P = 36}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 + 3x + 36 = 0$$

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$

Solution: As α, β are the roots of the equation $x^2 - 3x + 6 = 0$

$$a = 1, b = -3, c = 6$$

Therefore,

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-3)}{1} = 3 \Rightarrow \boxed{\alpha + \beta = 3}$$

$$\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6 \Rightarrow \boxed{\alpha\beta = 6}$$

$$\text{Sum of roots} = S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$$

$$S = (\alpha + \beta) \cdot \frac{1}{\alpha\beta}$$

$$S = 3 \times \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{S = \frac{1}{2}}$$

$$\text{Product of roots} = P = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)$$

$$P = \frac{1}{\alpha\beta} = \frac{1}{6} \quad \boxed{P = \frac{1}{6}}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{1}{2}x + \frac{1}{6} = 0$$

Multiplying by '6' on both side, we have

$$6x^2 - 3x + 1 = 0$$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution: As α, β are the roots of the equation $x^2 - 3x + 6 = 0$

$$a = 1, b = -3, c = 6$$

Therefore,

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-3)}{1} = 3 \Rightarrow \boxed{\alpha + \beta = 3}$$

$$\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6 \Rightarrow \boxed{\alpha\beta = 6}$$

$$\text{Sum of roots} = S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$S = \frac{\alpha^2 + \beta^2}{\beta\alpha}$$

$$S = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$S = \frac{(3)^2 - 2(6)}{6}$$

$$S = \frac{9 - 12}{6}$$

$$S = \frac{-3}{6}$$

$$\boxed{S = -\frac{1}{2}}$$

$$\text{Product of roots} = P = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 + \frac{1}{2}x + 1 = 0$$

Multiplying both sides by '2', we have

$$2x^2 + x + 2 = 0$$

(e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution: As α, β are the roots of the equation $x^2 - 3x + 6 = 0$

$$a = 1, b = -3, c = 6$$

Therefore,

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-3)}{1} = 3 \Rightarrow \boxed{\alpha + \beta = 3}$$

$$\alpha\beta = \frac{c}{a} = \frac{6}{1} = 6 \Rightarrow \boxed{\alpha\beta = 6}$$

$$\text{Sum of roots} = S = (\alpha + \beta) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$S = (\alpha + \beta) + \frac{\beta + \alpha}{\alpha\beta}$$

$$S = (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta}$$

$$S = 3 + \frac{3}{6}$$

$$S = 3 + \frac{1}{2}$$

$$S = \frac{6 + 1}{2}$$

$$\boxed{S = \frac{7}{2}}$$

$$\text{Product of roots} = P = (\alpha + \beta) \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$P = (\alpha + \beta) \left(\frac{\beta + \alpha}{\alpha\beta}\right)$$

$$P = (\alpha + \beta) \left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

$$P = 3 \left(\frac{3}{6}\right)$$

$$\boxed{P = \frac{3}{2}}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

Multiplying both sides by '2' we have

$$2x^2 - 7x + 3 = 0$$

Q.3 If α, β are the roots of the equation $x^2 + px + q = 0$. From equation whose roots are

$$(a). \alpha^2, \beta^2 \quad (b) \quad \frac{\alpha}{\beta}, \frac{\beta}{\alpha}$$

Solution:

Since α, β are the roots of the equation $x^2 + px + q = 0$

$$ax^2 + bx + c = 0$$

By comparing the coefficients of these equations, we have

$$a = 1, b = p, c = q$$

$$\alpha + \beta = \frac{-b}{a} = \frac{-p}{1} = -p \Rightarrow \boxed{\alpha + \beta = -p}$$

$$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q \Rightarrow \boxed{\alpha\beta = q}$$

$$(a) \quad \alpha^2, \beta^2$$

$$\text{Sum of roots} = S = \alpha^2 + \beta^2$$

$$S = (\alpha + \beta)^2 - 2\alpha\beta$$

$$S = (-p)^2 - 2q$$

$$S = p^2 - 2q$$

$$\text{Product of roots} = P = \alpha^2\beta^2$$

$$P = (\alpha\beta)^2$$

$$P = q^2$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - (p^2 - 2q)x + q^2 = 0$$

$$(b) \quad \frac{\alpha}{\beta}, \frac{\beta}{\alpha}$$

Solution:

Since α, β are the roots of the equation $x^2 + px + q = 0$

$$ax^2 + bx + c = 0$$

By comparing the coefficients of these equations, we have

$$a = 1, b = p, c = q$$

$$\alpha + \beta = \frac{-b}{a} = \frac{-p}{1} = -p \Rightarrow \boxed{\alpha + \beta = -p}$$

$$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q \Rightarrow \boxed{\alpha\beta = q}$$

$$\text{Sum of roots} = S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$S = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$S = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$S = \frac{(-p)^2 - 2(q)}{q}$$

$$S = \frac{p^2 - 2q}{q}$$

$$\text{Product of roots} = P = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$$

Multiplying by q

$$qx^2 - (p^2 - 2q)x + q = 0$$

Synthetic Division

Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial. In fact synthetic division is simply a shortcut of long division method.

Example 1:

Using synthetic division, divide the polynomial $P(x) = 5x^4 + x^3 - 3x$ by $x - 2$

Solution:

$$(5x^4 + x^3 - 3x) \div (x - 2)$$

From divisor, $x - a$, here $a = 2$

Now write the co-efficient of the dividend in row with zero as the co-efficient of the

missing powers of x in the descending order as shown below.

Dividend

$$P(x) = 5x^4 + 1x^3 + 0x^2 - 3x + 0x^0$$

Now write the co-efficient of x from dividend in a row and $a = 2$ on the left side.

	5	1	0	-3	0
2	↓	10	22	44	82
	5	11	22	41	82

- Write 5 the first co-efficient as it is in the row under horizontal line.
- Multiply 5 with 2 and write the result 10 under 1 write the sum of $1 + 10 = 11$ under the line.
- Multiply 11 with 2 and place the result 22 under 0. Add 0 and 22 and write the result 22 under the line.
- Multiply 22 with 2, place the result 44 under -3. Write 41 as the sum of 44 and -3 under the line.
- Multiply 41 with 2 and put the result 82 under 0. The sum of 0 and 82 is 82. In the highest power of x in dividend is 4, therefore, the highest power of x in quotient will be $4 - 1 = 3$. Thus

$$\text{Quotient} = Q(x) = 5x^3 + 11x^2 + 22x + 41$$

$$\text{and the Remainder} = R = 82$$

Use of synthetic division

- Find quotient and remainder, when a given polynomial is divided by a linear polynomial.

Example 2:

Using synthetic division, divide

$$P(x) = x^4 - x^2 + 15 \text{ by } x + 1$$

$$\text{Solution: } (x^4 - x^2 + 15) \div (x + 1)$$

$$(x^4 + 0x^3 - x^2 + 0x + 15) \div (x + 1)$$

$$\text{As } x + 1 = x - (-1), \text{ so, } a = -1$$

Now write the co-efficient of dividend in a row and $a = -1$ on the left side.

	1	0	-1	0	15
-1	↓	-1	1	0	0
	1	-1	0	0	15

$$\therefore \text{Quotient} = Q(x) = x^3 - x^2 + 0x + 0 = x^3 - x^2$$

$$\text{And Remainder} = 15$$

- Find the value (s) of unknown (s), if the zeros of a polynomial are given.

Example 3:

Using synthetic division, find the value of h . If the zero of polynomial

$$P(x) = 3x^2 + 4x - 7h \text{ is } 1$$

Solution:

$$P(x) = 3x^2 + 4x - 7h \text{ and its zero is } 1$$

Then by synthetic division.

	3	4	-7h
1	↓	3	7
	3	7	7-7h

$$\text{Remainder} = 7 - 7h$$

Since 1 is the zero of the polynomial, therefore, Remainder = 0, that is,

$$7 - 7h = 0$$

$$7 = 7h \Rightarrow \boxed{h = 1}$$

- Find the value (s) of unknown (s), if the factors of a polynomial are given.

Example 4:

Using synthetic division, find the values of l and m , if $x-1$ and $x+1$ are the factors of the polynomial $P(x) = x^3 + 3lx^2 + mx - 1$

Solution :

Since $x-1$ and $x+1$ are the factors of

$$P(x) = x^3 + 3lx^2 + mx - 1$$

Therefore, 1 and -1 are zeros of polynomial $P(x)$

Now by synthetic division

1	1	3l	-m	-1
1	↓	1	3l+1	3l+m+1
		1	3l+1	3l+m+1
		1	3l+1	3l+m+1

Since 1 is the zero of polynomial, therefore, remainder is zero, that is, $3l + m = 0$ (i)

And

-1	1	3l	m	-1
-1	↓	-1	-3l+1	3l-m-1
		1	3l-1	-3l+m+1
		1	3l-1	-3l+m+1

Since -1 is the zero of polynomial, therefore, remainder is zero, that is, $3l - m - 2 = 0$ (ii)

Adding equations (i) and (ii), we get

$$\begin{array}{r}
 3l + m = 0 \\
 3l - m - 2 = 0 \\
 \hline
 6l - 2 = 0
 \end{array}$$

$$6l - 2 = 0$$

$$6l = 2$$

$$\Rightarrow l = \frac{2}{6} = \frac{1}{3} \quad \boxed{l = \frac{1}{3}}$$

Put the value of l in equation (i)

$$3\left(\frac{1}{3}\right) + m = 0$$

$$1 + m = 0 \quad \Rightarrow \quad \boxed{m = -1}$$

$$\text{Thus } l = \frac{1}{3} \text{ and } m = -1$$

(d) Solve a cubic equation, if one root of the equation is given.

Examples 5:

Using synthetic division, solve the equation $3x^3 - 11x^2 + 5x + 3 = 0$ when 3 is the root of the equation.

Solution:

Since 3 is the root of the equation

$$3x^3 - 11x^2 + 5x + 3 = 0$$

Then by synthetic division, we get.

3	3	-11	5	3
3	↓	9	-6	-3
		3	-2	-1
		3	-2	-1

The depressed equation is $3x^2 - 2x - 1 = 0$

$$3x^2 - 2x - 1 = 0$$

$$3x(x-1) + 1(x-1) = 0$$

$$(x-1)(3x+1) = 0$$

$$\begin{array}{ll}
 \text{Either } x-1=0 & \text{or } 3x+1=0, \\
 x=1 & \text{or } 3x=-1 \\
 & \Rightarrow x = \frac{-1}{3}
 \end{array}$$

Hence 3, 1 and $\frac{-1}{3}$ are the roots of the given equation.

(e) Solve a biquadratic (quartic) equation, if two of the real roots of the equation are given.

Example 6:

By synthetic division solve the equation $x^4 - 49x^2 + 36x + 252 = 0$ having roots -2 and 6

Solution:

Since -2 and 6 are the roots of the given

$$\text{equation } x^4 - 49x^2 + 36x + 252 = 0$$

Then by synthetic division we get.

	1	0	-49	36	252
-2	↓	-2	4	90	-252
	1	-2	-45	126	0
6		6	24	-126	
	1	4	-21	0	

The depressed equation is

$$x^2 + 4x - 21 = 0$$

$$x^2 + 7x - 3x - 21 = 0$$

$$x(x+7) - 3(x+7) = 0$$

$$(x+7)(x-3) = 0$$

$$\begin{array}{ll}
 \text{Either } x+7=0 & \text{or } x-3=0 \\
 x=-7 & \text{or } x=3
 \end{array}$$

Thus -2, 6, -7 and 3 are the roots of the given equation.