

EXERCISE 3.3

Q1: Find a third proportional

(i) 6, 12

Solution:

Let third proportional is x then 6, 12, x

By proportion.

$$6 : 12 :: 12 : x$$

Product of Extremes = Product of Means

$$6(x) = 12 \times 12$$

$$6(x) = 144$$

$$x = \frac{144}{6} = 24$$

$$\boxed{x = 24}$$

(ii) $a^3, 3a^2$

Solution:

Let 3rd proportional is x then $a^3, 3a^2, x$

By proportion

$$a^3 : 3a^2 :: 3a^2 : x$$

Product of Extremes = Product of Means

$$xa^3 = (3a^2)(3a^2)$$

$$xa^3 = 9a^4$$

$$x = \frac{9a^4}{a^3}$$

$$x = 9a^{4-3}$$

$$\boxed{x = 9a}$$

(iii) $a^2 - b^2, a - b$

Solution:

Let 3rd proportional is x then $a^2 - b^2, a - b, x$

By proportion

$$a^2 - b^2 : a - b :: a - b : x$$

Product of Extremes = Product of Means

$$x(a^2 - b^2) = (a - b)(a - b)$$

$$x = \frac{(a - b)(a - b)}{a^2 - b^2}$$

$$x = \frac{(a - b)(\cancel{a - b})}{(a + b)(\cancel{a - b})}$$

$$\boxed{x = \frac{a - b}{a + b}}$$

(iv) $(x - y)^2, x^3 - y^3$

Solution:

Let 3rd proportional is " a " then $(x - y)^2, x^3 - y^3, a$

By proportion:

$$(x - y)^2 : x^3 - y^3 :: x^3 - y^3 : a$$

Product of Extremes = Product of Means

$$a(x - y)^2 = (x^3 - y^3)(x^3 - y^3)$$

$$a = \frac{(x^3 - y^3)^2}{(x - y)^2}$$

$$a = \frac{[(x - y)(x^2 + xy + y^2)]^2}{(x - y)^2}$$

$$a = \frac{(\cancel{x - y})^2 (x^2 + xy + y^2)^2}{(\cancel{x - y})^2}$$

$$\boxed{a = (x^2 + xy + y^2)^2}$$

(v) $(x + y)^2, x^2 - xy - 2y^2$

Solution:

Let 3rd proportional is " a "

Then $(x + y)^2, x^2 - xy - 2y^2, a$

By Proportion:

$$(x + y)^2 : x^2 - xy - 2y^2 :: x^2 - xy - 2y^2 : a$$

Product of Extremes = Product of Means:

$$a(x + y)^2 = (x^2 - xy - 2y^2)(x^2 - xy - 2y^2)$$

$$a = \frac{(x^2 - xy - 2y^2)^2}{(x + y)^2}$$

$$a = \frac{(x^2 - xy - y^2 - y^2)^2}{(x + y)^2}$$

$$a = \frac{(x^2 - y^2 - xy - y^2)^2}{(x + y)^2}$$

$$a = \frac{[(x + y)(x - y) - y(x + y)]^2}{(x + y)^2}$$

$$a = \frac{[(x + y)(x - y - y)]^2}{(x + y)^2}$$

$$= \frac{(\cancel{x + y})^2 (x - 2y)^2}{(\cancel{x + y})^2}$$

$$\boxed{a = (x - 2y)^2}$$

$$(vi) \quad \frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}$$

Solution:

Let 3rd proportional is x

$$\text{Then } \frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}, x$$

By proportion.

$$\frac{p^2 - q^2}{p^3 + q^3} : \frac{p - q}{p^2 - pq + q^2} :: \frac{p - q}{p^2 - pq + q^2} : x$$

Product of Extremes = Product of Means.

$$x \cdot \frac{p^2 - q^2}{p^3 + q^3} = \frac{p - q}{p^2 - pq + q^2} \times \frac{p - q}{p^2 - pq + q^2}$$

$$x \cdot \frac{p^2 - q^2}{p^3 + q^3} = \left[\frac{(p - q)}{(p^2 - pq + q^2)} \right]^2$$

$$x = \frac{(p - q)^2}{(p^2 - pq + q^2)^2} \times \frac{p^3 + q^3}{(p^2 - q^2)}$$

$$x = \frac{(\cancel{p - q})(p - q)}{(p^2 - pq + q^2)^{\cancel{2}}} \times \frac{(\cancel{p + q})(p^2 - pq + q^2)}{(\cancel{p + q})(\cancel{p - q})}$$

$$\boxed{x = \frac{p - q}{p^2 - pq + q^2}}$$

Q.2/ Find a fourth proportional

(i) 5, 8, 15

Solution:

Let 4th proportional is x then 5, 8, 15, x

By proportion

$$5 : 8 :: 15 : x$$

Product of Extremes = Product of Means

$$5(x) = 8(15)$$

$$x = \frac{8(\cancel{15}^3)}{\cancel{5}^1}$$

$$x = 8(3)$$

$$\boxed{x = 24}$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let 4th proportional is "a" then $4x^4, 2x^3, 18x^5, a$

By proportion

$$4x^4 : 2x^3 :: 18x^5 : a$$

Product of Extreme = Product of Means.

$$a(4x^4) = 2x^3(18x^5)$$

$$a = \frac{36x^8}{4x^4}$$

$$a = 9x^{8-4}$$

$$\boxed{a = 9x^4}$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let 4th proportional is x

then $15a^5b^6, 10a^2b^5, 21a^3b^3, x$

By proportion

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

Product of Extremes = Product of Means

$$x(15a^5b^6) = (10a^2b^5)(21a^3b^3)$$

$$x = \frac{14\cancel{21}0\cancel{x^8}b^8}{15\cancel{x^8}b^6}$$

$$x = 14b^{8-6}$$

$$\boxed{x = 14b^2}$$

(iv) $x^2 - 11x + 24; (x - 3), (5x^4 - 40x^3)$

Solution:

Let 4th proportional is "a"

$$x^2 - 11x + 24, (x - 3), (5x^4 - 40x^3), a$$

By proportion

$$x^2 - 11x + 24 : (x - 3) :: 5x^4 - 40x^3 : a$$

Product of Extremes = Product of Means

$$a(x^2 - 11x + 24) = (x - 3)(5x^4 - 40x^3)$$

$$a = \frac{(x - 3)(5x^4 - 40x^3)}{x^2 - 11x + 24}$$

$$a = \frac{(x - 3) \cdot 5x^3(x - 8)}{x^2 - 3x - 8x + 24}$$

$$a = \frac{5x^3(x - 3)(x - 8)}{x(x - 3) - 8(x - 3)}$$

$$a = \frac{5x^3(\cancel{x - 3})(\cancel{x - 8})}{(\cancel{x - 3})(\cancel{x - 8})}$$

$$\boxed{a = 5x^3}$$

(v) $p^3+q^3, p^2-q^2, p^2-pq+q^2$

Solution:

Let 4th proportional is x

$$p^3 + q^3, p^2 - q^2, p^2 - pq + q^2, x$$

By proportion

$$p^3+q^3, p^2-q^2, p^2-pq+q^2 : x$$

Product of Extremes = Product of Means.

$$x(p^3+q^3) = (p^2-q^2)(p^2-pq+q^2)$$

$$x = \frac{(p^2-q^2)(p^2-pq+q^2)}{p^3+q^3}$$

$$x = \frac{\cancel{(p+q)}(p-q)\cancel{(p^2-pq+q^2)}}{\cancel{(p+q)}\cancel{(p^2-pq+q^2)}}$$

$$\boxed{x = (p - q)}$$

(vi) $(p^2-q^2)(p^2+pq+q^2), p^3+q^3, p^3-q^3$

Solution:

Let 4th proportional is x.

Then $(p^2-q^2)(p^2+pq+q^2), p^3+q^3, p^3-q^3, x$

By proportion:

$$(p^2-q^2)(p^2+pq+q^2) : p^3+q^3 :: p^3-q^3 : x$$

Product of Extremes = Product of Means:

$$x(p^2-q^2)(p^2+pq+q^2) = (p^3+q^3)(p^3-q^3)$$

$$x = \frac{(p^3+q^3)(p^3-q^3)}{(p^2-q^2)(p^2+pq+q^2)}$$

$$x = \frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{(p-q)}\cancel{(p^2+pq+q^2)}}{\cancel{(p+q)}\cancel{(p-q)}\cancel{(p^2+pq+q^2)}}$$

$$\boxed{x = (p^2 - pq + q^2)}$$

Q.3: Find mean proportional:

(i) 20, 45

Solution:

Let mean proportional is m

then 20, m, 45

By proportion

$$20 : m :: m : 45$$

Product of Means = Product of Extremes

$$m.m = 20 \times 45$$

$$m^2 = 900$$

Taking square root

$$\sqrt{m^2} = \pm \sqrt{900}$$

$$\boxed{m = \pm 30}$$

(ii) $20x^3y^5, 5x^7y$

Solution:

Let mean proportional is m

then $20x^3y^5, m, 5x^7y$

By proportion,

$$20x^3y^5 : m :: m : 5x^7y$$

Product of Means = Product of Extremes

$$m.m = (20x^3y^5)(5x^7y)$$

$$m^2 = 100x^{10}y^6$$

Taking square root of both sides

$$\sqrt{m^2} = \pm \sqrt{100x^{10}y^6}$$

$$m = \pm \sqrt{100} \cdot \sqrt{x^{10}} \cdot \sqrt{y^6}$$

$$m = \pm \sqrt{100} \cdot \sqrt{x^{10}} \cdot \sqrt{y^6}$$

$$m = \pm 10x^{10 \times \frac{1}{2}} \cdot y^{6 \times \frac{1}{2}}$$

$$\boxed{m = \pm 10x^5y^3}$$

(iii) $15p^4qr^3, 135q^5r^7$

Solution:

Let mean proportional is m

then $15p^4qr^3, m, 135q^5r^7$

By proportion

$$15p^4qr^3 : m :: m : 135q^5r^7$$

Product of Means = Product of Extremes

$$m.m = (15p^4qr^3)(135q^5r^7)$$

$$m^2 = 2025p^4q^6r^{10}$$

Taking square root

$$\sqrt{m^2} = \pm \sqrt{2025p^4q^6r^{10}}$$

$$m = \pm \sqrt{2025} \cdot \sqrt{p^4} \cdot \sqrt{q^6} \cdot \sqrt{r^{10}}$$

$$m = \pm 45p^{4 \times \frac{1}{2}} \cdot q^{6 \times \frac{1}{2}} \cdot r^{10 \times \frac{1}{2}}$$

$$\boxed{m = \pm 45p^2 \cdot q^3 \cdot r^5}$$

$$(iv) \quad x^2 - y^2, \frac{x-y}{x+y}$$

Solution:

Let mean proportional is m.

$$\text{then } x^2 - y^2, m, \frac{x-y}{x+y}$$

By proportion

$$x^2 - y^2 : m :: m : \frac{x-y}{x+y}$$

Product of Means = Product of Extremes

$$m.m = (x^2 - y^2) \frac{(x-y)}{x+y}$$

$$m^2 = \frac{\cancel{(x+y)} (x-y)(x-y)}{\cancel{(x+y)}}$$

$$m^2 = (x-y)^2$$

Taking square root

$$\sqrt{m^2} = \pm \sqrt{(x-y)^2}$$

$$\boxed{m = \pm (x-y)}$$

Q.4 Find the values of the letter involved in the following continued proportions

$$(i) \quad 5, p, 45$$

Solution:

By continued proportion

$$5 : p :: p : 45$$

Product of Means = Product of Extremes

$$p.p = 5 \times 45$$

$$p^2 = 225$$

Taking square root of both sides

$$\sqrt{p^2} = \pm \sqrt{225}$$

$$\boxed{p = \pm 15}$$

$$(ii) \quad 8, x, 18$$

Solution:

By continued proportion

$$8 : x :: x : 18$$

Product of Means = Product of Extremes

$$x.x = 8 \times 18$$

$$x^2 = 144$$

Taking square root

$$\sqrt{x^2} = \pm \sqrt{144}$$

$$\boxed{x = \pm 12}$$

$$(iii) \quad 12, 3p-6, 27$$

Solution:

By continued proportion

$$12 : 3p-6 :: 3p-6 : 27$$

Product of Means = Product of Extremes.

$$(3p-6)(3p-6) = 12 \times 27$$

$$(3p-6)^2 = 324$$

Taking square root of both sides

$$\sqrt{(3p-6)^2} = \pm \sqrt{324}$$

$$3p-6 = \pm 18$$

$$3p-6 = 18 \quad \text{or} \quad 3p-6 = -18$$

$$3p = 18+6 \quad \text{or} \quad 3p = -18+6$$

$$3p = 24 \quad \text{or} \quad 3p = -12$$

$$p = \frac{24}{3} \quad \text{or} \quad p = \frac{-12}{3}$$

$$p = 8 \quad \text{or} \quad p = -4$$

$$(iv) \quad 7, m-3, 28$$

Solution:

By continued proportion

$$7 : m-3 :: m-3 : 28$$

Product of Means = Product of Extremes

$$(m-3)(m-3) = 7 \times 28$$

$$(m-3)^2 = 196$$

Taking square root of both sides

$$\sqrt{(m-3)^2} = \pm \sqrt{196}$$

$$m-3 = \pm 14$$

$$m-3 = 14 \quad \text{or} \quad m-3 = -14$$

$$m = 14+3 \quad \text{or} \quad m = -14+3$$

$$m = 17 \quad \text{or} \quad m = -11$$

Theorems on Proportions

If four quantities a , b , c and d form a proportion, then many other useful properties may be deduced by the properties of fractions.

(1) Theorem of Invertendo

If $a : b = c : d$, then $b : a = d : c$ is called invertendo theorem.

Example 1: If $3m : 2n = p : 2q$, then
 $2n : 3m = 2q : p$

Solution: Since $3m : 2n = p : 2q$

$$\therefore \frac{3m}{2n} = \frac{p}{2q}$$

By invertendo theorem

$$\frac{2n}{3m} = \frac{2q}{p}$$

$$\text{i.e. } 2n : 3m = 2q : p$$

(2) Theorem of Alternando

If $a : b = c : d$, then $a : c = b : d$ is called alternando theorem.

Example 2: If $3p + 1 : 2q = 5r : 7s$, then prove that $3p + 1 : 5r = 2q : 7s$

Solution:

Given that $3p + 1 : 2q = 5r : 7s$

$$\text{then } \frac{3p+1}{2q} = \frac{5r}{7s}$$

By alternando theorem

$$\frac{3p+1}{5r} = \frac{2q}{7s}$$

Thus, $3p + 1 : 5r = 2q : 7s$

(3) Theorem of Componendo

If $a : b = c : d$ then componendo theorem is written as.

$$(i) \quad a + b : b = c + d : d$$

$$\text{and } (ii) \quad a : a + b = c : c + d$$

Example 3: If $m + 3 : n = p : q - 2$, then prove that $m + n + 3 : n = p + q - 2 : q - 2$

Solution: Since $m + 3 : n = p : q - 2$

$$\therefore \frac{m+3}{n} = \frac{p}{q-2}$$

By componendo theorem

$$\frac{(m+3)+n}{n} = \frac{p+(q-2)}{q-2}$$

$$\text{or } \frac{m+n+3}{n} = \frac{p+q-2}{q-2}$$

$$\text{Thus } m+n+3 : n = p+q-2 : q-2$$

(4) Theorem of Dividendo

If $a : b = c : d$, then dividendo theorem is written as:

$$(i) \quad a - b : b = c - d : d$$

$$\text{and } (ii) \quad a : a - b = c : c - d$$

Example 4: If $m + 1 : n - 2 = 2p + 3 : 3q + 1$ Then $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

Solution:

Given that $m + 1 : n - 2 = 2p + 3 : 3q + 1$

$$\text{Then } \frac{m+1}{n-2} = \frac{2p+3}{3q+1}$$

By dividendo theorem

$$\frac{(m+1)-(n-2)}{n-2} = \frac{(2p+3)-(3q+1)}{3q+1}$$

$$\frac{(m+1-n+2)}{n-2} = \frac{2p+3-3q-1}{3q+1}$$

$$\frac{m-n+3}{n-2} = \frac{2p-3q+2}{3q+1}$$

$$\text{Thus } m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$$

(5) Theorem of componendo-dividendo

If $a : b = c : d$, then componendo-dividendo theorem is written as:

$$(i) \quad a + b : a - b = c + d : c - d$$

$$(ii) \quad a - b : a + b = c - d : c + d$$

Example 5: If $m : n = p : q$ then prove that $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Solution: Since $m : n = p : q$

$$\text{or } \frac{m}{n} = \frac{p}{q}$$

Multiplying both sides by $\frac{3}{7}$, we get

$$\frac{3m}{7n} = \frac{3p}{7q}$$

Then using componendo-dividendo theorem

$$\frac{3m+7n}{3m-7n} = \frac{3p+7q}{3p-7q}$$

$$\text{or } 3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$$

Example 6:

If $5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$, then show that $m : n = p : q$

Solution: Given that

$$5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$$

$$\text{Or } \frac{5m+3n}{5m-3n} = \frac{5p+3q}{5p-3q}$$

By componendo-dividendo theorem

$$\frac{(5m+3n)+(5m-3n)}{(5m+3n)-(5m-3n)} = \frac{(5p+3q)+(5p-3q)}{(5p+3q)-(5p-3q)}$$

$$\frac{\cancel{5m} + \cancel{3n} + \cancel{5m} - \cancel{3n}}{\cancel{5m} + 3n - \cancel{5m} + 3n} = \frac{\cancel{5p} + \cancel{3q} + \cancel{5p} - \cancel{3q}}{\cancel{5p} + 3q - \cancel{5p} + 3q}$$

$$\frac{10m}{6n} = \frac{10p}{6q}$$

Multiplying both sides by $\frac{6}{10}$

$$\frac{\cancel{10}}{\cancel{10}} \times \frac{\cancel{10}m}{\cancel{6}n} = \frac{\cancel{10}}{\cancel{10}} \times \frac{\cancel{10}p}{\cancel{6}q}$$

$$\frac{m}{n} = \frac{p}{q}$$

$$\text{i.e. } m : n = p : q$$

Example 7:

Using theorem of componendo - dividendo, find the value of

$$\frac{m+3p}{m-3p} + \frac{m+2q}{m-2q}, \text{ if } m = \frac{6pq}{p+q}$$

Solution: Since $m = \frac{6pq}{p+q}$ or

$$m = \frac{(3p)(2q)}{p+q} \dots\dots\dots(i)$$

$$\therefore \frac{m}{3p} = \frac{2q}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+3p}{m-3p} = \frac{p+3q}{q-p} \dots\dots\dots(ii)$$

Again from eq. (i), we have

$$\frac{m}{2q} = \frac{3p}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+2q}{m-2q} = \frac{3p+(p+q)}{3p-(p+q)} = \frac{3p+p+q}{3p-p-q}$$

$$\frac{m+2q}{m-2q} = \frac{4p+q}{2p-q} \dots\dots\dots(iii)$$

Adding (ii) and (iii)

$$\frac{m+3p}{m-3p} + \frac{m+2q}{m-2q} = \frac{p+3q}{q-p} + \frac{4p+q}{2p-q}$$

$$= -\frac{p+3q}{p-q} + \frac{4p+q}{2p-q}$$

$$= \frac{-(p+3q)(2p-q) + (p-q)(4p+q)}{(p-q)(2p-q)}$$

$$= \frac{-[2p^2 - pq + 6pq - 3q^2] + [4p^2 + pq - 4pq - q^2]}{(p-q)(2p-q)}$$

$$= \frac{-[2p^2 + 5pq - 3q^2] + [4p^2 - 3pq - q^2]}{(p-q)(2p-q)}$$

$$= \frac{-2p^2 - 5pq + 3q^2 + 4p^2 - 3pq - q^2}{(p-q)(2p-q)}$$

$$= \frac{2p^2 - 8pq + 2q^2}{(p-q)(2p-q)} = \frac{2(p^2 - 4pq + q^2)}{(p-q)(2p-q)}$$

Example 8:

Using theorem of componendo - dividendo solve the equation

$$\frac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \frac{4}{3}$$

Solution: Given equation is

$$\frac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \frac{4}{3}$$

By componendo-dividendo theorem

$$\frac{(\sqrt{x+3} + \sqrt{x-3}) + (\sqrt{x+3} - \sqrt{x-3})}{(\sqrt{x+3} + \sqrt{x-3}) - (\sqrt{x+3} - \sqrt{x-3})} = \frac{4+3}{4-3}$$

$$\frac{\cancel{\sqrt{x+3}} + \cancel{\sqrt{x-3}} + \sqrt{x+3} - \cancel{\sqrt{x-3}}}{\cancel{\sqrt{x+3}} + \sqrt{x-3} - \cancel{\sqrt{x+3}} + \sqrt{x-3}} = \frac{7}{1}$$

$$\frac{\cancel{2}\sqrt{x+3}}{\cancel{2}\sqrt{x-3}} = \frac{7}{1} \Rightarrow \sqrt{\frac{x+3}{x-3}} = 7$$

Squaring both sides

$$\left(\sqrt{\frac{x+3}{x-3}}\right)^2 = (7)^2$$

$$\frac{x+3}{x-3} = 49$$

$$x+3 = 49(x-3)$$

$$\Rightarrow x+3 = 49x-147$$

$$\Rightarrow x-49x = -147-3$$

$$-48x = -150$$

$$\Rightarrow 48x = 150$$

$$\Rightarrow x = \frac{150}{48} = \frac{25}{8} \Rightarrow \boxed{x = \frac{25}{8}}$$

Example 9:

Using componendo-dividendo theorem,

solve the equation $\frac{(x+3)^2 - (x-5)^2}{(x+3)^2 + (x-5)^2} = \frac{4}{5}$

Solution: Given equation is

$$\frac{(x+3)^2 - (x-5)^2}{(x+3)^2 + (x-5)^2} = \frac{4}{5}$$

By componendo-dividendo theorem

$$\frac{[(x+3)^2 - (x-5)^2] + [(x+3)^2 + (x-5)^2]}{[(x+3)^2 - (x-5)^2] - [(x+3)^2 + (x-5)^2]} = \frac{4+5}{4-5}$$

$$\frac{(x+3)^2 - \cancel{(x-5)^2} + (x+3)^2 + \cancel{(x-5)^2}}{\cancel{(x+3)^2} - (x-5)^2 - \cancel{(x+3)^2} - (x-5)^2} = \frac{4+5}{4-5}$$

$$\frac{\cancel{2}(x+3)^2}{\cancel{2}(x-5)^2} = \frac{9}{\cancel{1}}$$

$$\Rightarrow \left(\frac{x+3}{x-5}\right)^2 = (3)^2$$

Taking square root

$$\frac{x+3}{x-5} = \pm 3$$

$$\frac{x+3}{x-5} = 3 \quad \text{or} \quad \frac{x+3}{x-5} = -3$$

$$x+3 = 3(x-5) \quad \text{or} \quad x+3 = -3(x-5)$$

$$x+3 = 3x-15 \quad \text{or} \quad x+3 = -3x+15$$

$$3+15 = 3x-x \quad \text{or} \quad x+3x = 15-3$$

$$18 = 2x \quad \text{or} \quad 4x = 12$$

$$\frac{18}{2} = x \quad \text{or} \quad x = \frac{12}{4}$$

$$9 = x \quad \text{or} \quad x = 3$$

$$\Rightarrow x = 9$$

The solution set is $\{3, 9\}$