

EXERCISE 3.4

Q.1 Prove that $a : b = c : d$, if

$$(i) \quad \frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

$$(ii) \quad \frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$

$$(iii) \quad \frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$$

$$(iv) \quad \frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$$

$$(v) \quad pa+qb : pa-qb = pc+qd : pc-qd$$

$$(vi) \quad \frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$$

$$(vii) \quad \frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$$

$$(viii) \quad \frac{a^2+b^2}{a^2-b^2} = \frac{ac+bd}{ac-bd}$$

Solutions:

$$(i) \quad \frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

Solution:

$$\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

By componendo – dividendo theorem

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)} = \frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a + \cancel{5b} + 4a - \cancel{5b}}{\cancel{4a} + 5b - \cancel{4a} + 5b} = \frac{4c + \cancel{5d} + 4c - \cancel{5d}}{\cancel{4c} + 5d - \cancel{4c} + 5d}$$

$$\frac{8a}{10b} = \frac{8c}{10d}$$

By multiplying both sides by $\frac{10}{8}$

$$\frac{\cancel{10}}{\cancel{8}} \frac{\cancel{8}}{\cancel{10}} \frac{a}{b} = \frac{\cancel{10}}{\cancel{8}} \frac{\cancel{8}}{\cancel{10}} \frac{c}{d}$$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow a : b = c : d \text{ Hence proved}$$

$$(ii) \quad \frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$

Solution:

$$\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$

By componendo – dividendo theorem

$$\frac{(2a+9b)+(2a-9b)}{(2a+9b)-(2a-9b)} = \frac{(2c+9d)+(2c-9d)}{(2c+9d)-(2c-9d)}$$

$$\frac{2a+9b+2a-9b}{2a+9b-2a+9b} = \frac{2c+9d+2c-9d}{2c+9d-2c+9d}$$

$$\frac{4a}{18b} = \frac{4c}{18d}$$

Multiplying both sides by $\frac{18}{4}$

$$\frac{18}{4} \times \frac{4a}{18b} = \frac{4c}{18d} \cdot \frac{18}{4}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a : b = c : d$$

Hence proved

$$(iii) \quad \frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$$

Solution:

$$\frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$$

By componendo – dividendo theorem

$$\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)} = \frac{(c^3+d^3)+(c^3-d^3)}{(c^3+d^3)-(c^3-d^3)}$$

$$\frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2} = \frac{c^3+d^3+c^3-d^3}{c^3+d^3-c^3+d^3}$$

$$\frac{2ac^2}{2bd^2} = \frac{2c^3}{2d^3}$$

$$\frac{ac^2}{bd^2} = \frac{c.c^2}{d.d^2}$$

Multiplying both sides by $\frac{d^2}{c^2}$

$$\frac{a \cancel{d^2}}{b \cancel{d^2}} \times \frac{\cancel{d^2}}{\cancel{c^2}} = \frac{c \cancel{d^2}}{d \cancel{d^2}} \times \frac{\cancel{d^2}}{\cancel{c^2}}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a : b = c : d \quad \text{Hence proved}$$

$$(iv) \quad \frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$$

Solution:

$$\frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$$

By componendo – dividendo theorem

$$\frac{(a^2c+b^2d)+(a^2c-b^2d)}{(a^2c+b^2d)-(a^2c-b^2d)} = \frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}$$

$$\frac{a^2c+b^2d+a^2c-b^2d}{a^2c+b^2d-a^2c+b^2d} = \frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2}$$

$$\frac{2a^2c}{2b^2d} = \frac{2ac^2}{2bd^2}$$

$$\frac{a.ac}{b.bd} = \frac{ac.c}{bd.d}$$

Multiplying both sides by $\frac{bd}{ac}$

$$\frac{a \cancel{ac}}{b \cancel{bd}} \times \frac{\cancel{bd}}{\cancel{ac}} = \frac{\cancel{bd}}{\cancel{ac}} \cdot \frac{\cancel{ac}.c}{\cancel{bd}.d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow a : b = c : d \quad \text{Hence proved}$$

$$(v) \quad pa+qb : pa-qb = pc+qd : pc-qd$$

Solution:

$$pa+qb : pa-qb = pc+qd : pc-qd$$

OR

$$\frac{pa+qb}{pa-qb} = \frac{pc+qd}{pc-qd}$$

By componendo – dividendo theorem

$$\frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)} = \frac{(pc+qd)+(pc-qd)}{(pc+qd)-(pc-qd)}$$

$$\frac{pa+qb+pa-qb}{pa+qb-pa+qb} = \frac{pc+qd+pc-qd}{pc+qd-pc+qd}$$

$$(vi) \quad \frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}$$

Solution:

Let 3rd proportional is x

$$\text{Then } \frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}, x$$

By proportion.

$$\frac{p^2 - q^2}{p^3 + q^3} : \frac{p - q}{p^2 - pq + q^2} :: \frac{p - q}{p^2 - pq + q^2} : x$$

Product of Extremes = Product of Means.

$$x \cdot \frac{p^2 - q^2}{p^3 + q^3} = \frac{p - q}{p^2 - pq + q^2} \times \frac{p - q}{p^2 - pq + q^2}$$

$$x \cdot \frac{p^2 - q^2}{p^3 + q^3} = \left[\frac{(p - q)}{(p^2 - pq + q^2)} \right]^2$$

$$x = \frac{(p - q)^2}{(p^2 - pq + q^2)^2} \times \frac{p^3 + q^3}{(p^2 - q^2)}$$

$$x = \frac{(\cancel{p - q})(p - q)}{(p^2 - pq + q^2)^{\cancel{2}}} \times \frac{(\cancel{p + q})(p^2 - pq + q^2)}{(\cancel{p + q})(\cancel{p - q})}$$

$$\boxed{x = \frac{p - q}{p^2 - pq + q^2}}$$

Q.2/ Find a fourth proportional

(i) 5, 8, 15

Solution:

Let 4th proportional is x then 5, 8, 15, x

By proportion

$$5 : 8 :: 15 : x$$

Product of Extremes = Product of Means

$$5(x) = 8(15)$$

$$x = \frac{8(\cancel{15}^3)}{\cancel{5}^1}$$

$$x = 8(3)$$

$$\boxed{x = 24}$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let 4th proportional is "a" then $4x^4, 2x^3, 18x^5, a$

By proportion

$$4x^4 : 2x^3 :: 18x^5 : a$$

Product of Extreme = Product of Means.

$$a(4x^4) = 2x^3(18x^5)$$

$$a = \frac{36x^8}{4x^4}$$

$$a = 9x^{8-4}$$

$$\boxed{a = 9x^4}$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let 4th proportional is x

then $15a^5b^6, 10a^2b^5, 21a^3b^3, x$

By proportion

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

Product of Extremes = Product of Means

$$x(15a^5b^6) = (10a^2b^5)(21a^3b^3)$$

$$x = \frac{14\cancel{21}0\cancel{x^8}b^8}{15\cancel{x^5}b^6}$$

$$x = 14b^{8-6}$$

$$\boxed{x = 14b^2}$$

(iv) $x^2 - 11x + 24; (x - 3), (5x^4 - 40x^3)$

Solution:

Let 4th proportional is "a"

$$x^2 - 11x + 24, (x - 3), (5x^4 - 40x^3), a$$

By proportion

$$x^2 - 11x + 24 : (x - 3) :: 5x^4 - 40x^3 : a$$

Product of Extremes = Product of Means

$$a(x^2 - 11x + 24) = (x - 3)(5x^4 - 40x^3)$$

$$a = \frac{(x - 3)(5x^4 - 40x^3)}{x^2 - 11x + 24}$$

$$a = \frac{(x - 3) \cdot 5x^3(x - 8)}{x^2 - 3x - 8x + 24}$$

$$a = \frac{5x^3(x - 3)(x - 8)}{x(x - 3) - 8(x - 3)}$$

$$a = \frac{5x^3(\cancel{x - 3})(\cancel{x - 8})}{(\cancel{x - 3})(\cancel{x - 8})}$$

$$\boxed{a = 5x^3}$$

$$\frac{(a^2 + b^2) + (a^2 - b^2)}{(a^2 + b^2) - (a^2 - b^2)} = \frac{(ac + bd) + (ac - bd)}{(ac + bd) - (ac - bd)}$$

$$\frac{\cancel{a^2} + \cancel{b^2} + \cancel{a^2} - \cancel{b^2}}{\cancel{a^2} + b^2 - \cancel{a^2} + b^2} = \frac{\cancel{ac} + \cancel{bd} + \cancel{ac} - \cancel{bd}}{\cancel{ac} + bd - \cancel{ac} + bd}$$

$$\frac{\cancel{2}a^2}{\cancel{2}b^2} = \frac{\cancel{2}ac}{\cancel{2}bd}$$

$$\frac{a.a}{b.a} = \frac{ac}{bd}$$

Multiplying both sides by " $\frac{b}{a}$ "

$$\frac{\cancel{b}}{\cancel{a}} \cdot \frac{\cancel{a}.a}{\cancel{b}.a} = \frac{\cancel{b}}{\cancel{a}} \cdot \frac{\cancel{a}.c}{\cancel{b}.d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$\Rightarrow a : b = c : d$ Hence proved

Q.2 Use componendo–dividendo theorem to find the values of the following.

(i) Find the value of

$$\frac{x + 2y}{x - 2y} + \frac{x + 2z}{x - 2z}, \text{ if } x = \frac{4yz}{y + z}$$

Solution: $x = \frac{4yz}{y + z}$ (i)

Dividing the equation (i) by "2y"

$$\frac{x}{2y} = \frac{4yz}{(y + z).2y}$$

$$\frac{x}{2y} = \frac{2z}{y + z}$$

By componendo – dividendo theorem

$$\frac{x + 2y}{x - 2y} = \frac{2z + (y + z)}{2z - (y + z)}$$

$$\frac{x + 2y}{x - 2y} = \frac{2z + y + z}{2z - y - z}$$

$$\frac{x + 2y}{x - 2y} = \frac{3z + y}{z - y} \text{ (ii)}$$

Now dividing the eq (i) by "2z"

$$\frac{x}{2z} = \frac{4yz}{(y + z).2z}$$

$$\frac{x}{2z} = \frac{2y}{y + z}$$

By componendo – dividendo theorem

$$\frac{x + 2z}{x - 2z} = \frac{2y + (y + z)}{2y - (y + z)}$$

$$\frac{x + 2z}{x - 2z} = \frac{2y + y + z}{2y - y - z}$$

$$\frac{x + 2z}{x - 2z} = \frac{3y + z}{y - z} \text{ (iii)}$$

Adding eq. (ii) and (iii)

$$\begin{aligned} \frac{x + 2y}{x - 2y} + \frac{x + 2z}{x - 2z} &= \frac{3z + y}{z - y} + \frac{3y + z}{y - z} \\ &= \frac{3z + y}{-1(y - z)} + \frac{3y + z}{y - z} \\ &= \frac{-1(3z + y)}{y - z} + \frac{3y + z}{y - z} \\ &= \frac{-3z - y + 3y + z}{y - z} \\ &= \frac{2y - 2z}{y - z} \\ &= \frac{2(\cancel{y} - \cancel{z})}{(\cancel{y} - \cancel{z})} \end{aligned}$$

$$\frac{x + 2y}{x - 2y} + \frac{x + 2z}{x - 2z} = 2$$

(ii) Find the value of

$$\frac{m + 5n}{m - 5n} + \frac{m + 5p}{m - 5p}, \text{ if } m = \frac{10np}{n + p}$$

Solution: $m = \frac{10np}{n + p}$ (i)

Dividing equation (i) by "5n"

$$\frac{m}{5n} = \frac{10np}{(n + p)5n}$$

$$\frac{m}{5n} = \frac{2p}{n + p}$$

By componendo – dividendo theorem

$$\frac{m+5n}{m-5n} = \frac{2p+(n+p)}{2p-(n+p)}$$

$$\frac{m+5n}{m-5n} = \frac{2p+n+p}{2p-n-p}$$

$$\frac{m+5n}{m-5n} = \frac{3p+n}{p-n} \dots\dots\dots (ii)$$

Now, dividing equation (i) by “5p”

$$\frac{m}{5p} = \frac{10np}{(n+p)5p}$$

$$\frac{m}{5p} = \frac{2n}{n+p}$$

By componendo– dividendo theorem

$$\frac{m+5p}{m-5p} = \frac{2n+(n+p)}{2n-(n+p)}$$

$$\frac{m+5p}{m-5p} = \frac{2n+n+p}{2n-n-p}$$

$$\frac{m+5p}{m-5p} = \frac{3n+p}{n-p} \dots\dots\dots (iii)$$

Adding equation (ii) and (iii)

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = \frac{3p+n}{p-n} + \frac{3n+p}{n-p}$$

$$= \frac{3p+n}{-1(n-p)} + \frac{3n+p}{n-p}$$

$$= \frac{-(3p+n)}{n-p} + \frac{3n+p}{n-p}$$

$$= \frac{-3p-n+3n+p}{n-p}$$

$$= \frac{2n-2p}{n-p}$$

$$= \frac{2(\cancel{n-p})}{(\cancel{n-p})}$$

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = 2$$

(iii) Find the value of

$$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} \text{ if } x = \frac{12ab}{a-b}$$

$$\text{Solution: } x = \frac{12ab}{a-b} \dots\dots\dots (i)$$

Dividing equation (i) by 6a

$$\frac{x}{6a} = \frac{12ab}{(a-b).6a}$$

$$\frac{x}{6a} = \frac{2b}{a-b}$$

By componendo– dividendo theorem

$$\frac{x+6a}{x-6a} = \frac{2b+(a-b)}{2b-(a-b)}$$

$$\frac{x+6a}{x-6a} = \frac{2b+a-b}{2b-a+b}$$

$$\frac{x+6a}{x-6a} = \frac{a+b}{3b-a}$$

By invertendo theorem

$$\frac{x-6a}{x+6a} = \frac{3b-a}{a+b} \dots\dots\dots (i)$$

Now, dividing the equation (i) by 6b.

$$\frac{x}{6b} = \frac{12ab}{(a-b).6b}$$

$$\frac{x}{6b} = \frac{2a}{a-b}$$

By componendo–dividendo theorem

$$\frac{x+6b}{x-6b} = \frac{2a+(a-b)}{2a-(a-b)}$$

$$\frac{x+6b}{x-6b} = \frac{2a+a-b}{2a-a+b}$$

$$\frac{x+6b}{x-6b} = \frac{3a-b}{a+b} \dots\dots\dots (ii)$$

Subtracting equation (iii) from (ii)

$$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} = \frac{3b-a}{a+b} - \frac{3a-b}{a+b}$$

$$= \frac{(3b-a)-(3a-b)}{a+b}$$

$$= \frac{3b-a-3a+b}{a+b}$$

$$= \frac{4b - 4a}{a + b}$$

$$= \frac{4(b - a)}{(a + b)}$$

(iv) Find the value of

$$\frac{x - 3y}{x + 3y} - \frac{x + 3z}{x - 3z}, \text{ if } x = \frac{3yz}{y - z}$$

Solution: $x = \frac{3yz}{y - z}$ (i)

Dividing equation (i) by "3y"

$$\frac{x}{3y} = \frac{3yz}{(y - z) \cdot 3y}$$

$$\frac{x}{3y} = \frac{z}{y - z}$$

By componendo – dividendo theorem

$$\frac{x + 3y}{x - 3y} = \frac{z + (y - z)}{z - (y - z)}$$

$$\frac{x + 3y}{x - 3y} = \frac{\cancel{z} + y - \cancel{z}}{z - y + z}$$

$$\frac{x + 3y}{x - 3y} = \frac{y}{2z - y}$$

By invertendo theorem

$$\frac{x - 3y}{x + 3y} = \frac{2z - y}{y} \text{ (ii)}$$

Now dividing equation (i) by "3z"

$$\frac{x}{3z} = \frac{3yz}{(y - z) \cdot 3z}$$

$$\frac{x}{3z} = \frac{y}{y - z}$$

By componendo – dividendo theorem

$$\frac{x + 3z}{x - 3z} = \frac{y + (y - z)}{y - (y - z)}$$

$$\frac{x + 3z}{x - 3z} = \frac{y + y - z}{\cancel{y} - \cancel{y} + z}$$

$$\frac{x + 3z}{x - 3z} = \frac{2y - z}{z} \text{ (iii)}$$

Subtracting equation (iii) from (ii)

$$\frac{x - 3y}{x + 3y} - \frac{x + 3z}{x - 3z} = \frac{2z - y}{y} - \frac{2y - z}{z}$$

$$= \frac{z(2z - y) - y(2y - z)}{yz}$$

$$= \frac{2z^2 - \cancel{yz} - 2y^2 + \cancel{yz}}{yz}$$

$$= \frac{2z^2 - 2y^2}{yz} = \frac{2(z^2 - y^2)}{yz}$$

(v) Find the value of

$$\frac{s - 3p}{s + 3p} + \frac{s + 3q}{s - 3q}, \text{ if } s = \frac{6pq}{p - q}$$

Solution:

$$s = \frac{6pq}{p - q} \text{ (i)}$$

Dividing equation (i) by "3P"

$$\frac{s}{3p} = \frac{6pq}{(p - q) \cdot 3p}$$

$$\frac{s}{3p} = \frac{2q}{p - q}$$

By componendo – dividendo theorem

$$\frac{s + 3p}{s - 3p} = \frac{2q + (p - q)}{2q - (p - q)}$$

$$\frac{s + 3p}{s - 3p} = \frac{2q + p - q}{2q - p + q}$$

$$\frac{s + 3p}{s - 3p} = \frac{p + q}{3q - p}$$

By invertendo theorem

$$\frac{s - 3p}{s + 3p} = \frac{3q - p}{p + q} \text{ (ii)}$$

Now dividing equation (i) by "3q"

$$\frac{s}{3q} = \frac{\cancel{2p} \cancel{q}}{(p - q) \cdot \cancel{p} \cancel{q}}$$

$$\frac{s}{3q} = \frac{2p}{p - q}$$

By componendo – dividendo theorem

$$\frac{s + 3q}{s - 3q} = \frac{2p + (p - q)}{2p - (p - q)}$$

$$\frac{s + 3q}{s - 3q} = \frac{2p + p - q}{2p - p + q}$$

$$\frac{s+3q}{s-3q} = \frac{3p-q}{p+q} \dots\dots\dots(iii)$$

Adding equation (ii) and (iii)

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = \frac{3q-p}{p+q} + \frac{3p-q}{p+q}$$

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = \frac{3q-p+3p-q}{(p+q)}$$

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = \frac{2p+2q}{(p+q)}$$

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = \frac{2(\cancel{p+q})}{(\cancel{p+q})}$$

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q} = 2$$

(vi) Solve $\frac{(x-2)^2 - (x-4)^2}{(x-2)^2 + (x-4)^2} = \frac{12}{13}$

Solution:

$$\frac{(x-2)^2 - (x-4)^2}{(x-2)^2 + (x-4)^2} = \frac{12}{13}$$

By componendo-dividendo theorem

$$\frac{[(x-2)^2 - (x-4)^2] + [(x-2)^2 + (x-4)^2]}{[(x-2)^2 - (x-4)^2] - [(x-2)^2 + (x-4)^2]} = \frac{12+13}{12-13}$$

$$\frac{(x-2)^2 - \cancel{(x-4)^2} + (x-2)^2 + \cancel{(x-4)^2}}{\cancel{(x-2)^2} - (x-4)^2 - \cancel{(x-2)^2} - (x-4)^2} = \frac{25}{-1}$$

$$\frac{\cancel{2}(x-2)^2}{\cancel{2}(x-4)^2} = \frac{25}{-1}$$

$$\left(\frac{x-2}{x-4}\right)^2 = 25$$

Taking square root of both side

$$\sqrt{\left(\frac{x-2}{x-4}\right)^2} = \pm\sqrt{25}$$

$$\frac{x-2}{x-4} = \pm 5$$

$$\frac{x-2}{x-4} = 5 \quad \text{or} \quad \frac{x-2}{x-4} = -5$$

$$x-2 = 5(x-4) \quad \text{or} \quad x-2 = -5(x-4)$$

$$x-2 = 5x-20 \quad \text{or} \quad x-2 = -5x+20$$

$$20-2 = 5x-x \quad \text{or} \quad x+5x = 20+2$$

$$18 = 4x \quad \text{or} \quad 6x = 22$$

$$\frac{\cancel{9}18}{\cancel{2}4} = x \quad \text{or} \quad x = \frac{22}{6}$$

$$x = \frac{9}{2} \quad \text{or} \quad x = \frac{11}{3}$$

$$S.S = \left\{ \frac{9}{2}, \frac{11}{3} \right\}$$

(vii) Solve $\frac{\sqrt{x^2+2} + \sqrt{x^2-2}}{\sqrt{x^2+2} - \sqrt{x^2-2}} = \frac{2}{1}$

Solution:

$$\frac{\sqrt{x^2+2} + \sqrt{x^2-2}}{\sqrt{x^2+2} - \sqrt{x^2-2}} = \frac{2}{1}$$

By componendo-dividendo theorem

$$\frac{(\sqrt{x^2+2} + \sqrt{x^2-2}) + (\sqrt{x^2+2} - \sqrt{x^2-2})}{(\sqrt{x^2+2} + \sqrt{x^2-2}) - (\sqrt{x^2+2} - \sqrt{x^2-2})} = \frac{2+1}{2-1}$$

$$\frac{\sqrt{x^2+2} + \cancel{\sqrt{x^2-2}} + \sqrt{x^2+2} - \cancel{\sqrt{x^2-2}}}{\cancel{\sqrt{x^2+2}} + \sqrt{x^2-2} - \cancel{\sqrt{x^2+2}} + \sqrt{x^2-2}} = \frac{3}{1}$$

$$\frac{\cancel{2}\sqrt{x^2+2}}{\cancel{2}\sqrt{x^2-2}} = 3$$

$$\sqrt{\frac{x^2+2}{x^2-2}} = 3$$

Taking square of both sides

$$\left(\sqrt{\frac{x^2+2}{x^2-2}}\right)^2 = (3)^2$$

$$\frac{x^2+2}{x^2-2} = 9$$

$$x^2+2 = 9(x^2-2)$$

$$x^2+2 = 9x^2-18$$

$$2+18 = 9x^2-x^2$$

$$20 = 8x^2$$

$$x^2 = \frac{\cancel{5}20}{\cancel{2}8} \Rightarrow x^2 = \frac{5}{2}$$

Taking square root

$$\sqrt{x^2} = \pm\sqrt{\frac{5}{2}} \Rightarrow x = \pm\sqrt{\frac{5}{2}}$$

$$S.S = \left\{ \pm\sqrt{\frac{5}{2}} \right\}$$

(viii) Solve $\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}} = \frac{1}{3}$

Solution:

$$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}} = \frac{1}{3}$$

By componendo-dividendo theorem

$$\frac{(\sqrt{x^2+8p^2}-\sqrt{x^2-p^2})+(\sqrt{x^2+8p^2}+\sqrt{x^2-p^2})}{(\sqrt{x^2+8p^2}-\sqrt{x^2-p^2})-(\sqrt{x^2+8p^2}+\sqrt{x^2-p^2})} = \frac{1+3}{1-3}$$

$$\frac{\sqrt{x^2+8p^2}-\cancel{\sqrt{x^2-p^2}}+\sqrt{x^2+8p^2}+\cancel{\sqrt{x^2-p^2}}}{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}-\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}} = \frac{4}{-2}$$

$$\frac{\cancel{2}\sqrt{x^2+8p^2}}{\cancel{2}\sqrt{x^2-p^2}} = \cancel{2}$$

$$\sqrt{\frac{x^2+8p^2}{x^2-p^2}} = 2$$

Taking square of both sides

$$\left(\sqrt{\frac{x^2+8p^2}{x^2-p^2}}\right)^2 = (2)^2$$

$$\frac{x^2+8p^2}{x^2-p^2} = 4$$

$$x^2+8p^2 = 4(x^2-p^2)$$

$$x^2+8p^2 = 4x^2-4p^2$$

$$8p^2+4p^2 = 4x^2-x^2$$

$$12p^2 = 3x^2$$

$$\Rightarrow 3x^2 = 12p^2$$

$$x^2 = \frac{12p^2}{3}$$

$$x^2 = 4p^2$$

Taking square root

$$\sqrt{x^2} = \pm\sqrt{4p^2}$$

$$x = \pm 2p$$

$$S.S = \{\pm 2p\}$$

(ix) Solve $\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} = \frac{13}{14}$

Solution:

$$\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} = \frac{13}{14}$$

By componendo-dividendo theorem

$$\frac{[(x+5)^3-(x-3)^3]+[(x+5)^3+(x-3)^3]}{[(x+5)^3-(x-3)^3]-[(x+5)^3+(x-3)^3]} = \frac{13+14}{13-14}$$

$$\frac{(x+5)^3-\cancel{(x-3)^3}+(x+5)^3+\cancel{(x-3)^3}}{\cancel{(x+5)^3}-(x-3)^3-\cancel{(x+5)^3}-(x-3)^3} = \frac{27}{-1}$$

$$\frac{\cancel{2}(x+5)^3}{\cancel{2}(x-3)^3} = \cancel{2}27$$

$$\left(\frac{x+5}{x-3}\right)^3 = 27$$

Taking cube root

$$\sqrt[3]{\left(\frac{x+5}{x-3}\right)^3} = \sqrt[3]{27}$$

$$\left(\frac{x+5}{x-3}\right)^{\cancel{3} \times \frac{1}{\cancel{3}}} = (3)^{\cancel{3} \times \frac{1}{\cancel{3}}}$$

$$\frac{x+5}{x-3} = 3$$

$$(x+5) = 3(x-3)$$

$$x+5 = 3x-9$$

$$5+9 = 3x-x$$

$$14 = 2x$$

$$\frac{14}{2} = x$$

$$7 = x$$

$$\Rightarrow x = 7$$

$$S.S = \{7\}$$

Joint variation

A combination of direct and inverse variations of one or more than one variables forms joint variation. If a variable y varies directly as x and varies inversely as z .

Then $y \propto x$ and $y \propto \frac{1}{z}$

In joint variation, we write it as

$y \propto \frac{x}{z}$ i.e $y = k \frac{x}{z}$

Where $k \neq 0$ is the constant of variation.

For example, by Newton's law of gravitation, if one body attracts another with a force F that varies directly as the product of their masses (m_1), (m_2) and inversely as the square of the distance (d) between them.

i.e $F \propto \frac{m_1 m_2}{d^2}$ or $F = k \frac{m_1 m_2}{d^2}$

Problems related to joint variation

Example 1: If y varies jointly as x^2 and z and $y = 6$ when $x = 4$, $z = 9$. Write y as a function x and z and determine the value of y , when $x = -8$ and $z = 12$.

Solution: Since y varies jointly as x^2 and z , therefore

$y \propto x^2 z$
i.e $y = kx^2 z$ (i)

To find k ,

Put $y = 6$, $x = 4$, $z = 9$ in equation (i)

$6 = k(4)^2(9)$

$\frac{6}{16 \times 9} = k$

$\Rightarrow k = \frac{6}{144} = \frac{1}{24}$

Put $k = \frac{1}{24}$ in equation (i)

$y = \frac{1}{24} x^2 z$ (ii)

To find y ,

Now put $x = -8$, $z = 12$ in equation (i)

$y = \frac{1}{24} (-8)^2 (12)$

$y = \frac{1}{2} (64) \Rightarrow \boxed{y = 32}$

Example 2:

p varies jointly as q and r^2 and inversely as s and t^2 , $p = 40$, when $q = 8$, $r = 5$, $S=3$, $t =2$. Find p in terms of q , r , s and t . Also find the value of p when $q = -2$, $r = 4$, $s = 3$ and $t = -1$.

Solution: Given that $p \propto \frac{qr^2}{st^2}$

$p = k \frac{qr^2}{st^2}$ (i)

To find k ,

Put $p = 40$, $q = 8$, $r = 5$, $s = 3$ and $t = 2$ in equation (i)

$40 = k \frac{(8)(5)^2}{3(2)^2}$

$40 = k \frac{8(25)}{3(4)}$

$\frac{40 \times 12}{8(25)} = k$

$\Rightarrow k = \frac{12}{5}$

Put $k = \frac{12}{5}$ in equation (i)

$p = \frac{12}{5} \frac{qr^2}{st^2}$ (ii)

To find P ,

Put $q = -2$, $r = 4$, $s = 3$ and $t = -1$ in equation (i)

$P = \frac{12}{5} \times \frac{(-2)(4)^2}{(3)(-1)^2}$

$P = \frac{12}{5} \cdot \frac{(-2)(16)}{3}$

$p = \frac{(-8) \times (16)}{5}$

$\boxed{p = -\frac{128}{5}}$