

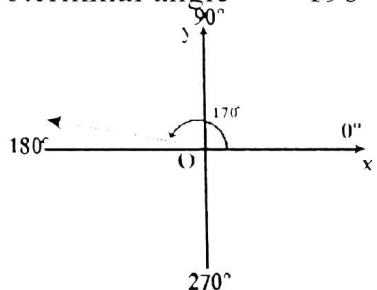
EXERCISE 7.3

Q.1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle coterminal with each given angle:

(i) 170°

Positive coterminal angle $= 360^\circ + 170^\circ$
 $= 530^\circ$

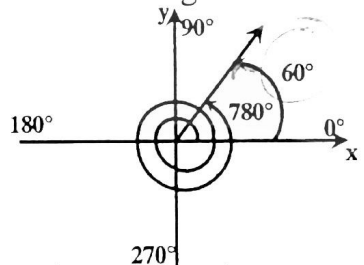
Negative coterminal angle $= -190^\circ$



(ii) 780°

Positive coterminal angle $780^\circ + 2(360^\circ) - 60^\circ = 60^\circ$

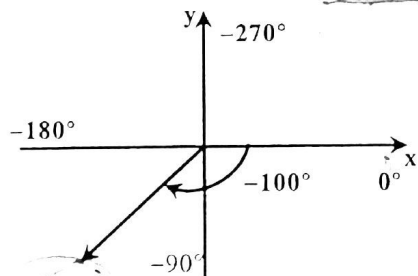
Negative coterminal angle $= -300^\circ$



(iii) -100°

Positive coterminal angle $= 260^\circ$

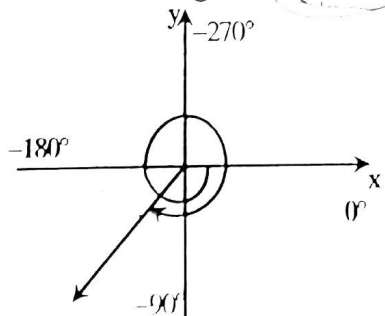
Negative coterminal angle $= -360^\circ - 100^\circ$
 $= -460^\circ$



(iv) -500°

Positive coterminal angle $= 220^\circ$

Negative coterminal angle $= -140^\circ$



Q.2. Identify closest quadrantal angles between which the following angles lie.

(i) 156°

Ans: 90° and 180°

(ii) 318°

Ans: 270° and 360°

(iii) 572°

Ans: 540° and 630°

(iv) -330°

Ans: 0° and 90°

Q.3. Write the closest quadrantal angles between which the angles lie. Write your answer in radian measure.

(i) $\frac{\pi}{3}$

Ans: 0 and $\frac{\pi}{2}$

(ii) $\frac{3\pi}{4}$

Ans: $\frac{\pi}{2}$ and π

(iii) $\frac{-\pi}{4}$

Ans: 0 and $-\frac{\pi}{2}$

(iv) $-\frac{3\pi}{4}$

Ans: $-\frac{\pi}{2}$ and $-\pi$

Q.4. In which quadrant θ lies, when

(i) $\sin \theta > 0, \tan \theta < 0$

Ans: II quadrant

(ii) $\cos \theta < 0, \sin \theta < 0$

Ans: III quadrant

(iii) $\sec \theta > 0, \sin \theta < 0$

Ans: IV quadrant

(iv) $\cos \theta < 0, \tan \theta < 0$

Ans: II quadrant

(v) $\operatorname{cosec} \theta > 0, \cos \theta > 0$

Ans: I quadrant

(vi) $\sin \theta < 0, \sec \theta < 0$

Ans: III quadrant

Q.5. Fill in the blanks:

- (i) $\cos(-150^\circ) = \underline{\hspace{2cm}} \cos 150^\circ$
- (ii) $\sin(-310^\circ) = \underline{\hspace{2cm}} \sin 310^\circ$
- (iii) $\tan(-210^\circ) = \underline{\hspace{2cm}} \tan 210^\circ$
- (iv) $\cot(-45^\circ) = \underline{\hspace{2cm}} \cot 45^\circ$
- (v) $\sec(-60^\circ) = \underline{\hspace{2cm}} \sec 60^\circ$
- (vi) $\operatorname{cosec}(-137^\circ) = \underline{\hspace{2cm}} \operatorname{cosec} 137^\circ$

Answers:

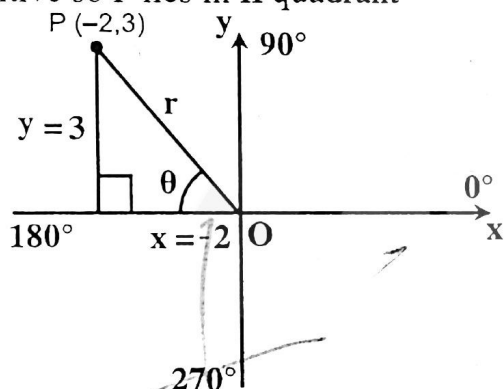
- (i) +ve (ii) -ve (iii) -ve
- (iv) -ve (v) +ve (vi) -ve

Q.6. The given point P lies on the terminal side of θ . Find quadrant of θ and all six trigonometric ratios.

(i) $(-2, 3)$

Solution: $P(x, y) = P(-2, 3)$

As x-coordinate is negative and y-coordinate is positive so P lies in II quadrant



The point P can be shown in II quadrant.

By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-2)^2 + (3)^2}$$

$$r = \sqrt{4+9}$$

$$r = \sqrt{13}$$

$$\text{Now, } \sin \theta = \frac{y}{r} = \frac{3}{\sqrt{13}}$$

$$\operatorname{cosec} \theta = \frac{r}{y} = \frac{\sqrt{13}}{3}$$

$$\cos \theta = \frac{x}{r} = -\frac{2}{\sqrt{13}}$$

$$\sec \theta = \frac{r}{x} = -\frac{\sqrt{13}}{2}$$

$$\tan \theta = \frac{y}{x} = -\frac{3}{2}$$

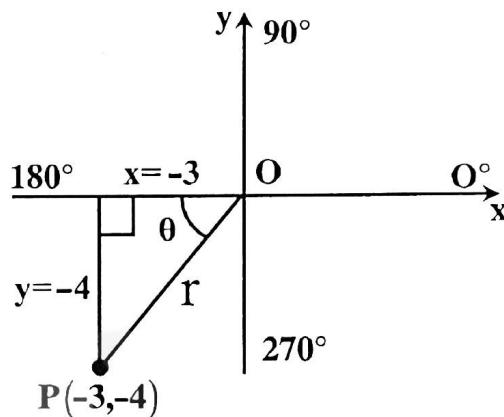
$$\cot \theta = \frac{x}{y} = -\frac{2}{3}$$

(ii) $(-3, -4)$

Solution: $P(x, y) = P(-3, -4)$

As x and y both coordinates are negative, so 'P' lies in III quadrant.

The point P can be shown in III quadrant.



By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-3)^2 + (-4)^2}$$

$$r = \sqrt{9+16}$$

$$r = \sqrt{25}$$

$$r = 5$$

$$\text{Now } \sin \theta = \frac{y}{r} = \frac{-4}{5}$$

$$\operatorname{cosec} \theta = \frac{r}{y} = \frac{-5}{4}$$

$$\cos \theta = \frac{x}{r} = \frac{-3}{5}$$

$$\sec \theta = \frac{r}{x} = \frac{-5}{3}$$

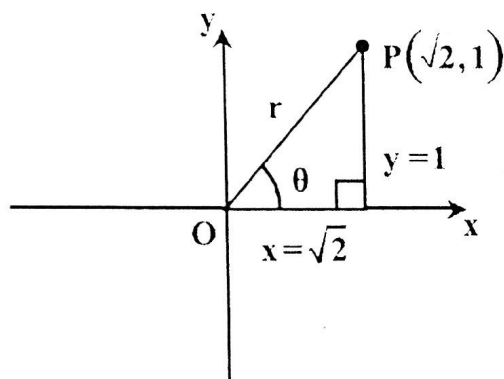
$$\tan \theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$$

$$\cot \theta = \frac{x}{y} = \frac{-3}{-4} = \frac{3}{4}$$

(iii) $(\sqrt{2}, 1)$

Solution: $P(x, y) = P(\sqrt{2}, 1)$

As x and y both coordinates are positive, so P lies in I quadrant.



The point 'P' can be shown in quadrant I.

By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(\sqrt{2})^2 + (1)^2}$$

$$r = \sqrt{2+1}$$

$$r = \sqrt{3}$$

$$\text{Now } \sin\theta = \frac{y}{r} = \frac{1}{\sqrt{3}}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$$

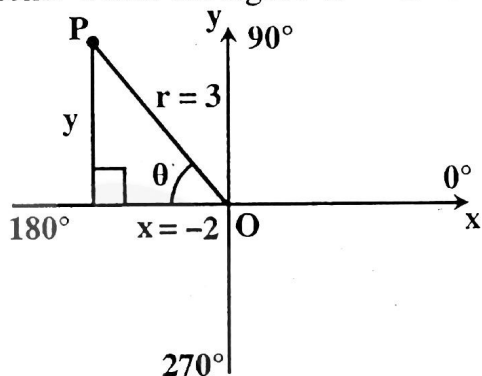
$$\cot\theta = \frac{x}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Q.7. If $\cos\theta = \frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.

Solution:

As $\cos\theta = \frac{-2}{3}$ and θ is in quadrant II, so

we complete the figure according to conditions. From the figure $x = -2$ and $r = 3$



By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{(3)^2 - (-2)^2}$$

$$y = \sqrt{9-4}$$

$$y = \sqrt{5}$$

Now

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{3}{\sqrt{5}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{-3}{2}$$

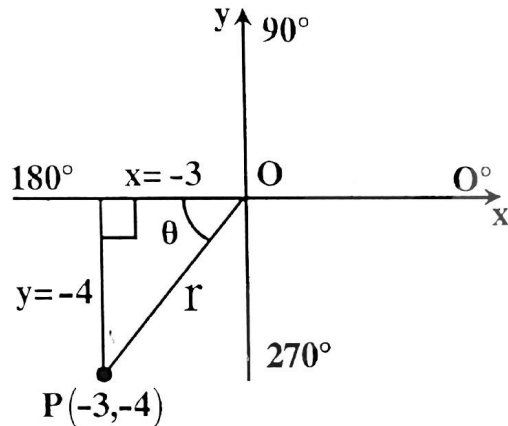
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{5}}{2}$$

$$\cot\theta = \frac{x}{y} = \frac{-2}{\sqrt{5}}$$

Q.8. If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ .

Solution:

As $\tan\theta = \frac{4}{3}$ and $\sin\theta$ is -ve, which is possible in quadrant III only. We complete the figure.



From the figure $x = -3$ and $y = -4$

By pathagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-3)^2 + (-4)^2}$$

$$r = \sqrt{9+16}$$

$$r = \sqrt{25}$$

$$r = 5$$

Now

$$\sin\theta = \frac{y}{r} = \frac{-4}{5}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{-5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{-5}{3}$$

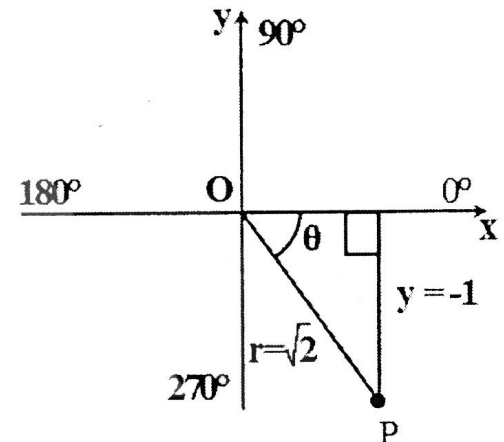
$$\tan\theta = \frac{y}{x} = \frac{4}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{3}{4}$$

Q.9. If $\sin\theta = -\frac{1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant III, find the values of $\tan\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$.

Solution:

As $\sin\theta = \frac{-1}{\sqrt{2}}$ and terminal side of angle is not in III quadrant, so it lies in quadrant IV.



From the figure $y = -1$ and $r = \sqrt{2}$

By Pathagorean theorem

$$r^2 = x^2 + y^2$$

$$x^2 = r^2 - y^2$$

$$x = \sqrt{r^2 - y^2}$$

$$x = \sqrt{(\sqrt{2})^2 - (-1)^2}$$

$$x = \sqrt{2-1}$$

$$x = \sqrt{1}$$

$$x = 1$$

Now

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

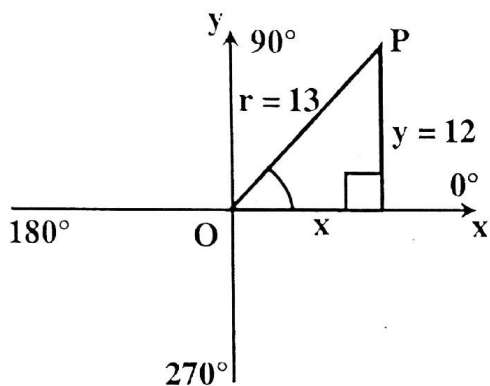
$$\sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

Q.10. If $\operatorname{cosec}\theta = \frac{13}{12}$ and $\sec\theta > 0$, find the remaining trigonometric functions.

Solution:

As $\operatorname{cosec}\theta = \frac{13}{12}$ and also $\sec\theta$ is +ve, which is only possible in quadrant I.



From the figure $y = 12$ and $r = 13$

By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$x^2 = r^2 - y^2$$

$$x = \sqrt{r^2 - y^2}$$

$$x = \sqrt{(13)^2 - (12)^2}$$

$$x = \sqrt{169 - 144}$$

$$x = \sqrt{25}$$

$$x = 5$$

Now

$$\sin\theta = \frac{y}{r} = \frac{12}{13}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{13}{12}$$

$$\cos\theta = \frac{x}{r} = \frac{5}{13}$$

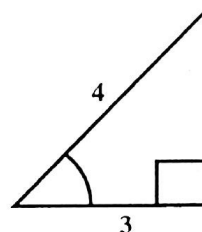
$$\sec\theta = \frac{r}{x} = \frac{13}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{12}{5}$$

$$\cot\theta = \frac{x}{y} = \frac{5}{12}$$

Q.11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.

(i)



From the figure Hypotenuse = 4 and Base = 3
By Pythagorean theorem we can find perpendicular

$$(\text{Per.})^2 + (\text{Base})^2 = (\text{Hyp.})^2$$

$$(\text{Per.})^2 + (3)^2 = (4)^2$$

$$(\text{Per.})^2 = 16 - 9$$

$$(\text{Per.})^2 = 7$$

$$\text{Perpendicular} = \sqrt{7}$$

$$\text{Now } \sin\theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{\sqrt{7}}{4}$$

$$\operatorname{cosec}\theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{4}{\sqrt{7}}$$

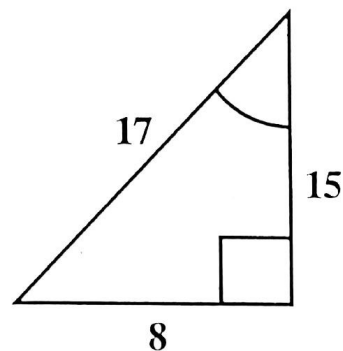
$$\cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{3}{4}$$

$$\sec\theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{4}{3}$$

$$\tan\theta = \frac{\text{Per.}}{\text{Base}} = \frac{\sqrt{7}}{3}$$

$$\cot\theta = \frac{\text{Base}}{\text{Per.}} = \frac{3}{\sqrt{7}}$$

(ii)



From the figure

Hypotenuse = 17

Perpendicular = 8

Base = 15

Now

$$\sin \theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{8}{17}$$

$$\operatorname{cosec} \theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{17}{8}$$

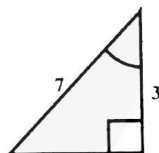
$$\cos \theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{15}{17}$$

$$\sec \theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{17}{15}$$

$$\tan = \frac{\text{Per.}}{\text{Base}} = \frac{8}{15}$$

$$\cot \theta = \frac{\text{Base}}{\text{Per.}} = \frac{15}{8}$$

(iii)



From the figure Hypotenuse = 7, Base = 3
We can find perpendicular by Pythagorean theorem.

$$(\text{Base})^2 + (\text{Per.})^2 = (\text{Hyp.})^2$$

$$(\text{Per.})^2 + (3)^2 = (7)^2$$

$$(\text{Per.})^2 = 49 - 9$$

$$(\text{Per.})^2 = 40$$

$$\text{Per.} = \sqrt{40}$$

$$\text{Per.} = \sqrt{4 \times 10}$$

$$\text{Per.} = 2\sqrt{10}$$

$$\text{Now, } \sin \theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{2\sqrt{10}}{7}$$

$$\operatorname{cosec} \theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{7}{2\sqrt{10}}$$

$$\cos \theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{3}{7}$$

$$\sec \theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{7}{3}$$

$$\tan \theta = \frac{\text{Per.}}{\text{Base}} = \frac{2\sqrt{10}}{3}$$

$$\cot \theta = \frac{\text{Base}}{\text{Per.}} = \frac{3}{2\sqrt{10}}$$

Q.12. Find the values of the trigonometric functions. Do not use trigonometric table or calculator.

Solution:

We know that $2k\pi + \theta = \theta$, where $k \in \mathbb{Z}$

(i) **$\tan 30^\circ$**

$$30^\circ = 30 \frac{\pi}{180} \text{ radian} = \frac{\pi}{6} \text{ radian}$$

$$\tan 30^\circ = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(ii) **$\tan 330^\circ$**

$$\tan 330^\circ = \tan (360^\circ - 30^\circ)$$

$$= \tan 2\pi - \frac{\pi}{6}$$

$$= \tan -\frac{\pi}{6}$$

$$= -\tan \frac{\pi}{6}$$

$$= -\frac{1}{\sqrt{3}}$$

(iii) **$\sec 330^\circ$**

$$\sec 330^\circ = \sec (360^\circ - 30^\circ)$$

$$= \sec 2\pi - \frac{\pi}{6}$$

$$= \sec -\frac{\pi}{6}$$

$$= \sec \frac{\pi}{6}$$

$$= \frac{2}{\sqrt{3}}$$

(iv) **$\cot \frac{\pi}{4}$**

$$\cot \frac{\pi}{4} = 1$$

(v) **$\cos \frac{2\pi}{3}$**

$$\cos \frac{2\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$(vi) \quad \operatorname{cosec} \frac{2\pi}{3}$$

$$\operatorname{cosec} \frac{2\pi}{3} = \operatorname{cosec} \frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

$$(vii) \quad \cos(-450^\circ)$$

$$\cos(-450^\circ) = \cos(-360^\circ - 90^\circ)$$

$$= \cos -2\pi - \frac{\pi}{2}$$

$$= \cos 2(-1)\pi - \frac{\pi}{2}$$

$$= \cos -\frac{\pi}{2}$$

$$= \cos \frac{\pi}{2} = 0$$

$$(viii) \quad \tan(-9\pi)$$

$$\tan(-9\pi) = \tan(-8\pi - \pi)$$

$$= \tan[2(-4)\pi - \pi]$$

$$= \tan(-\pi)$$

$$= -\tan \pi$$

$$= -(0) = 0$$

$$(ix) \quad \cos \frac{-5\pi}{6}$$

$$\cos \frac{-5\pi}{6} = -\cos \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2}$$

$$(x) \quad \sin 7\frac{\pi}{6}$$

$$\sin 7\frac{\pi}{6} = \sin \pi + \frac{\pi}{6}$$

$$= -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$(xi) \quad \cot \frac{7\pi}{6}$$

$$\cot \frac{7\pi}{6} = \cot \pi + \frac{\pi}{6}$$

$$= \cot \frac{\pi}{6} = \sqrt{3}$$

$$(xii) \quad \cos 225^\circ$$

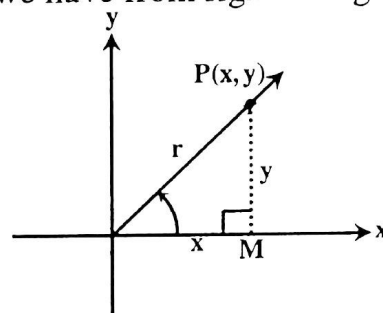
$$\cos 225^\circ = \cos(180^\circ + 45^\circ)$$

$$= \cos \pi + \frac{\pi}{4}$$

$$= -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Trigonometric Identities

Consider an angle $\angle MOP = \theta$ radian in standard position. Let point P (x, y) be on the terminal side of the angle. By Pythagorean theorem, we have from right triangle OMP.



$$(\overline{OM})^2 = (\overline{MP})^2 + (\overline{OP})^2$$

$$x^2 + y^2 = r^2 \dots\dots\dots(i)$$

Dividing both sides by r^2 , we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \left(\begin{array}{l} \sin \theta = \frac{y}{r} \\ \cos \theta = \frac{x}{r} \\ \tan \theta = \frac{y}{x} \end{array} \right)$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

$$\boxed{\cos^2 \theta + \sin^2 \theta = 1} \dots\dots\dots(i)$$

Dividing (i) by x^2 , we have

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{r^2}{x^2} = 1$$

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

As $\tan \theta = \frac{y}{x}$ and $\sec \theta = \frac{r}{x}$

$$1 + (\tan \theta)^2 = (\sec \theta)^2$$

$$1 + \tan^2 \theta = (\sec \theta)^2$$

$$1 + \tan^2 \theta = \sec^2 \theta \dots\dots\dots(ii)$$

or $\sec^2 \theta - \tan^2 \theta = 1$

Again dividing both sides of (i) by y^2 , we get

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} + \frac{r^2}{y^2} = 1$$

$$\frac{x^2}{y^2} = 1 + \frac{r^2}{y^2}$$

$$\cot \theta = \frac{x}{y} \text{ and } \operatorname{cosec} \theta = \frac{r}{y}$$

$$(\cot \theta)^2 + 1 = (\operatorname{cosec} \theta)^2$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \dots\dots\dots(iii)$$

or $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

The identities (1), (2) and (3) are also known as Pythagorean Identities.

Example 1: Verify that $\cot\theta\sec\theta = \operatorname{cosec}\theta$

Solution:

$$\begin{aligned}\text{L.H.S} &= \cot\theta\sec\theta \\&= \frac{\cos\theta}{\sin\theta} \frac{1}{\cos\theta} \\&= \frac{1}{\sin\theta} \\&= \operatorname{cosec}\theta \\ \text{L.H.S} &= \text{R.H.S}\end{aligned}$$

Example 2: Verify that

$$\tan^4\theta + \tan^2\theta = \tan^2\theta \sec^2\theta$$

Solution:

$$\begin{aligned}\text{L.H.S} &= \tan^4\theta + \tan^2\theta \\&= \tan^2\theta (\tan^2\theta + 1) \quad \tan^2\theta + 1 = \sec^2\theta \\&= \tan^2\theta \sec^2\theta \\ \text{L.H.S} &= \text{L.H.S}\end{aligned}$$

Example 3:

Show that $\frac{\cot^2}{\operatorname{cosec} - 1} = \operatorname{cosec} \alpha + 1$

Solution:

$$\begin{aligned}\text{L.H.S} &= \frac{\cot^2\alpha}{\operatorname{cosec}\alpha - 1} & \operatorname{cosec}^2\theta - \cot^2\theta &= 1 \\& & \cot^2\theta &= \operatorname{cosec}^2\theta - 1 \\&= \frac{(\operatorname{cosec}^2\alpha - 1)}{\operatorname{cosec}\alpha - 1} \\&= \frac{(\operatorname{cosec}\alpha - 1)(\operatorname{cosec}\alpha + 1)}{(\operatorname{cosec}\alpha - 1)} \\&= \operatorname{cosec}\alpha + 1 \\ \text{L.H.S} &= \text{R.H.S}\end{aligned}$$

Example 4:

Express the trigonometric functions in terms of $\tan\theta$.

Solution:

- By using reciprocal identity, we can express $\cot\theta$ in terms of $\tan\theta$.

$$\text{i.e } \cot\theta = \frac{1}{\tan\theta}$$

- By solving the identity $1 + \tan^2\theta = \sec^2\theta$

We have expressed $\sec\theta$ in terms of $\tan\theta$

$$\sec\theta = \pm \sqrt{\tan^2\theta + 1}$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\cos\theta = \frac{1}{\pm \sqrt{\tan^2\theta + 1}}$$

Because

$$\sin\theta = \tan\theta \cos\theta, \text{ we have}$$

$$\sin\theta = \tan\theta \frac{1}{\pm \sqrt{\tan^2\theta + 1}}$$

$$\sin\theta = \frac{\tan\theta}{\pm \sqrt{\tan^2\theta + 1}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\pm \sqrt{\tan^2\theta + 1}}{\tan\theta}$$