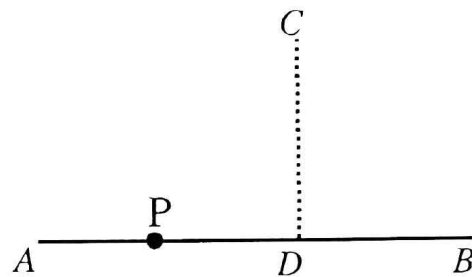


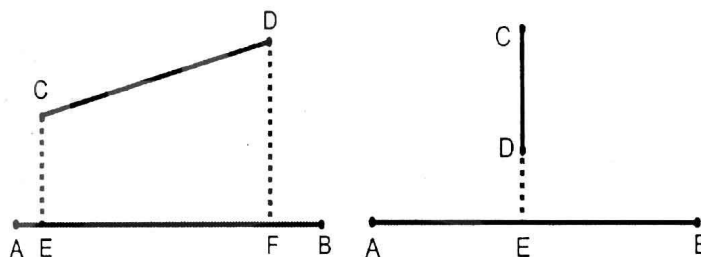
Projection of a Point:

The projection of a given point on a line segment is the foot of $\perp$ drawn from the point on that line segment. If $\overline{CD} \perp \overline{AB}$, then evidently D is the foot of perpendicular CD from the point C on the line segment AB. However projection of a point p lying on the $\overline{AB}$ is the point itself.



Projection of a Line Segment:

The projection of a line segment $\overline{CD}$ on a line segment $\overline{AB}$ is the portion $\overline{EF}$ of the latter intercepted between foots of the perpendiculars drawn from C and D. However projection of a vertical line segment $\overline{CD}$ on a line segment $\overline{AB}$ is a point on $\overline{AB}$ which is of zero dimension.



THEOREM 1

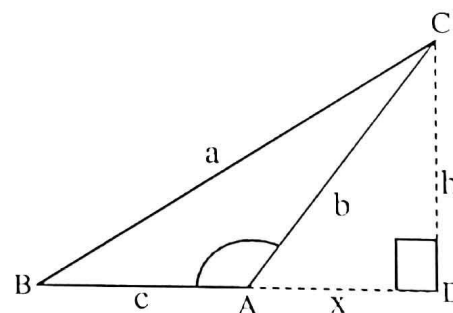
In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given: ABC is a triangle having an obtuse angle BAC at A. Draw $\overline{CD}$ perpendicular on $\overline{BA}$ produced, so that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.

Take $m\overline{BC} = a$, $m\overline{CA} = b$, $m\overline{AB} = c$, $m\overline{AD} = x$ and $m\overline{CD} = h$

To prove: $(m\overline{BC})^2 + (m\overline{AC})^2 - (m\overline{AB})^2 = 2(m\overline{AB})(m\overline{AD})$

$$\text{i.e., } a^2 = b^2 + c^2 + 2cx$$



Proof:

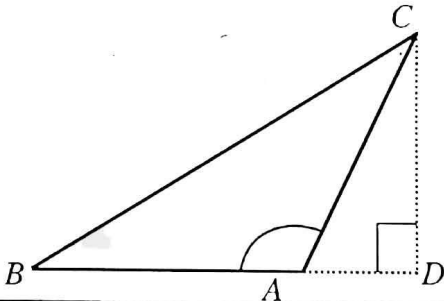
Statements	Reasons
In $\angle \text{rt } \triangle CDA$, $m\angle CDA = 90^\circ$ $(m\overline{AC})^2 + (m\overline{AD})^2 = (m\overline{CD})^2$ or $b^2 = x^2 + h^2 \dots\dots\dots(i)$ In $\angle \text{rt } \triangle CDB$, $m\angle CDB = 90^\circ$ $(m\overline{BC})^2 + (m\overline{BD})^2 = (m\overline{CD})^2$ or $a^2 = (c + x)^2 + h^2$ $a^2 = c^2 + 2cx + x^2 + h^2 \dots\dots\dots(ii)$ Hence $a^2 = c^2 + 2cx + b^2$ i.e., $a^2 = b^2 + c^2 + 2cx$ or $(m\overline{BC})^2 + (m\overline{AC})^2 + (m\overline{AB})^2 = 2(m\overline{AB})(m\overline{AD})$	Given Pythagoras Theorem Given Pythagoras Theorem $m\overline{BD} + m\overline{BA} = m\overline{AD}$ Using (i) and (ii)

Example: In a $\triangle ABC$ with obtuse angle at A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced and $m\overline{AC} = m\overline{AB}$ then prove that $(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD})$

Given: In a $\triangle ABC$, $m\angle A$ is obtuse $m\overline{AC} = m\overline{AB}$
and $\overline{CD}$ being altitude on $\overline{BA}$ produced.

To prove: $(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD})$

Proof: In a $\triangle ABC$, having obtuse angle BAC at A.



Statements	Reasons
$(m\overline{BC})^2 + (m\overline{BA})^2 + (m\overline{AC})^2 = 2(m\overline{BA})(m\overline{AD})$ $= (m\overline{AB})^2 + (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$ $= 2(m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$ $(m\overline{BC})^2 + 2m\overline{AB}(m\overline{AB} = m\overline{AD})$ $(m\overline{BC})^2 = 2(m\overline{AB})(m\overline{BD})$	By theorem 1 $m\overline{AC} = m\overline{AB}$ (Given) Taking $2(m\overline{AB})$ as common On the line segment BD, Point A is between B and D.

EXERCISE 8.1

Q. 1 Given $\overline{AC} = 1\text{cm}$, $\overline{BC} = 2\text{cm}$, $m\angle C = 120^\circ$

Compute the length $\overline{AB}$ and the area of $\triangle ABC$.

Hint : $(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 + 2(\overline{AC})(\overline{CD})$

where $(\overline{CD}) = (\overline{BC}) \cos(180^\circ - C)$ (Use theorem I)

Solution:

Given: In a $\triangle ABC$ $\overline{AC} = 1\text{cm}$, $\overline{BC} = 2\text{cm}$, $m\angle C = 120^\circ$

To Find: (i) $\overline{AB}$ (ii) Area of $\triangle ABC$

Calculations:

(i) In obtuse angled triangle ABC , by theorem I

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 + 2(\overline{AC})(\overline{CD}) \dots\dots\dots (i)$$

In right angled $\triangle BCD$

$$\cos 60^\circ = \frac{\overline{CD}}{\overline{BC}}$$

$$\frac{1}{2} = \frac{\overline{CD}}{2}$$

$$\overline{CD} = 1\text{cm} \quad \boxed{\overline{CD} = 1\text{cm}}$$

Now putting the corresponding values in (i)

$$\begin{aligned} (\overline{AB})^2 &= (1\text{cm})^2 + (2\text{cm})^2 + 2(1\text{cm})(1\text{cm}) \\ &= 1\text{cm}^2 + 4\text{cm}^2 + 2\text{cm}^2 \\ &= 7\text{cm}^2 \end{aligned}$$

$$\sqrt{(\overline{AB})^2} = \sqrt{7\text{cm}^2} \quad \boxed{\overline{AB} = 2.645\text{ cm}}$$

(ii) Area of $\triangle ABC = \frac{1}{2} \text{ base} \times \text{Altitude}$

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} \overline{AC} \times \overline{BD} \\ &= \frac{1}{2} \times 1\text{cm} \times h \dots\dots\dots (ii) \end{aligned}$$

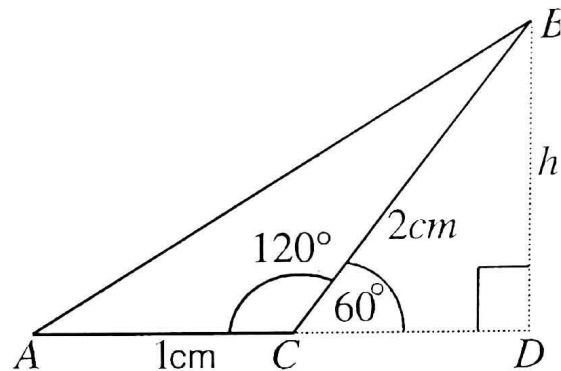
In right angled triangle BCD

By Pythagoras theorem

$$\begin{aligned} (2\text{cm})^2 &= (1\text{cm})^2 + (h)^2 \\ 4\text{cm}^2 &= 1\text{cm}^2 + h^2 \\ h^2 &= 3\text{cm}^2 \quad h = \sqrt{3}\text{ cm} \end{aligned}$$

Thus equation (ii) becomes

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} \times 1\text{cm} \times \sqrt{3}\text{ cm} \\ \text{Area of } \triangle ABC &= \frac{\sqrt{3}}{2}\text{cm}^2 \end{aligned}$$

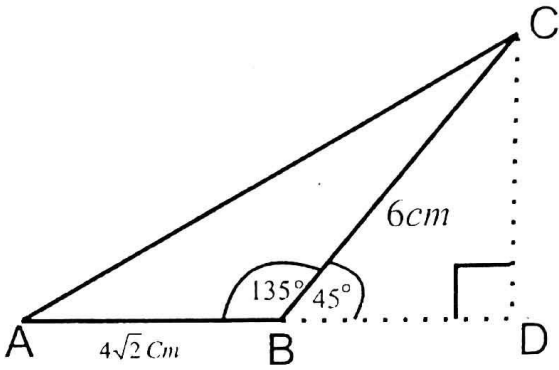


Q. 2 Find $m\overline{AC}$ if in $\triangle ABC$, $m\overline{BC} = 6\text{cm}$, $m\overline{AB} = 4\sqrt{2}\text{cm}$ and $m\angle ABC = 135^\circ$.

Solution:

Given:

$$\begin{aligned} m\overline{BC} &= 6\text{cm} \\ m\overline{AB} &= 4\sqrt{2}\text{cm} \\ m\angle ABC &= 135^\circ \end{aligned}$$



To Find: $m\overline{AC} = ?$

Calculation:

In obtuse angled triangle ABC, by theorem 1

$$(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 + 2(m\overline{AB})(m\overline{BD}).....(i)$$

In right angled $\triangle BCD$

$$\begin{aligned} \cos 45^\circ &= \frac{m\overline{BD}}{m\overline{BC}} \\ \frac{1}{\sqrt{2}} &= \frac{m\overline{BD}}{6\text{cm}} \\ m\overline{BD} &= \frac{6}{\sqrt{2}} \text{ cm} \end{aligned}$$

Now putting the corresponding values in equation (i) we get

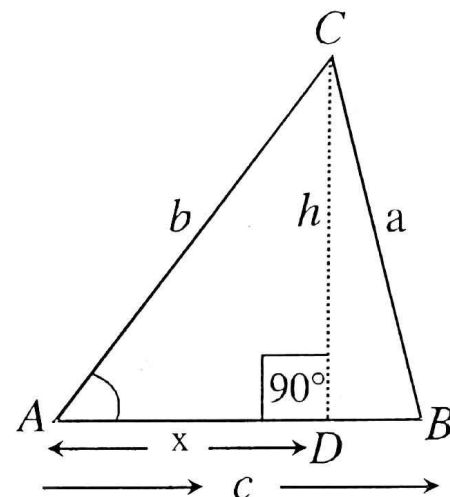
$$\begin{aligned} (m\overline{AC})^2 + (4\sqrt{2} \text{ cm})^2 + (6\text{cm})^2 &= 2(4\sqrt{2} \text{ cm}) \frac{6}{\sqrt{2}} \text{ cm} \\ &= 16(2 \text{ cm}^2) + 36\text{cm}^2 + 8\text{cm} (6\text{cm}) \\ &= 32 \text{ cm}^2 + 36 \text{ cm}^2 + 48 \text{ cm}^2 \\ &= 116 \text{ cm}^2 \end{aligned}$$

By taking square root of both sides, we get

$$\begin{aligned} \sqrt{(m\overline{AC})^2} &= \sqrt{116 \text{ cm}^2} = \sqrt{4 \times 29 \text{ cm}^2} \\ m\overline{AC} &= 2\sqrt{29} \text{ cm} \end{aligned}$$

THEOREM 2

In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.



Given: $\triangle ABC$ with an acute angle CAB at A .

Take $m\overline{BC} = a$ $m\overline{CA} = b$ and $m\overline{AB} = c$

Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also, $m\overline{AD} = x$ and $m\overline{CD} = h$

To prove: $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$

i.e., $a^2 = b^2 + c^2 - 2cx$

Proof:

Statements	Reasons
In $\angle rt \triangle CDA$	
$m\angle CDA = 90^\circ$	Given
$(m\overline{AC})^2 + (m\overline{AD})^2 = (m\overline{CD})^2$	Pythagoras theorem
i.e., $b^2 = x^2 + h^2$(i)	
In $\angle rt \triangle CDB$,	
$m\angle CDB = 90^\circ$	Given
$(m\overline{BC})^2 + (m\overline{BD})^2 = (m\overline{CD})^2$	Pythagoras theorem
$a^2 = (c - x)^2 + h^2$	From the figure $m\overline{BD} = (c - x)$
or $a^2 = c^2 - 2cx + x^2 + h^2$(ii)	
$a^2 = c^2 - 2cx + b^2$	Using (i) and (ii)
Hence, $a^2 = b^2 + c^2 - 2cx$	
i.e., $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$	

THEOREM 3

(APOLLONIUS' THEOREM)

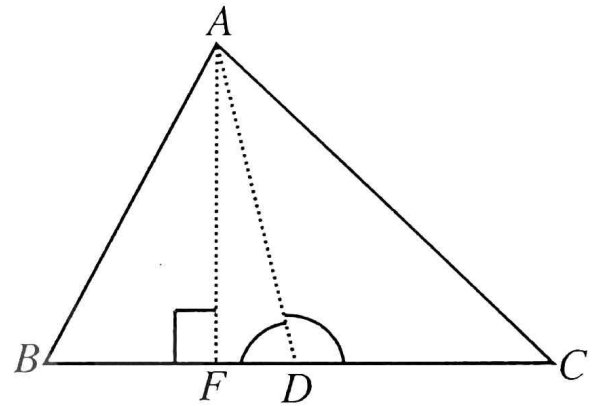
In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

Given: In a $\triangle ABC$, the median $\overline{AD}$ bisects $\overline{BC}$.

i.e., $m\overline{BD} = m\overline{CD}$

To prove: $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$

Construction: Draw $\overline{AF} \perp \overline{BC}$



Proof:

Statements	Reasons
In $\triangle ADB$ Since $\angle ADB$ is acute at D	
$\therefore (m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2 - 2(m\overline{BD})(m\overline{FD}) \dots\dots(i)$	Using theorem 2
Now in $\triangle ADC$ since $\angle ADC$ is obtuse at D	
$\therefore (m\overline{AC})^2 = (m\overline{CD})^2 + (m\overline{AD})^2 - 2(m\overline{CD})(m\overline{FD}) \dots\dots(ii)$	Using theorem 1
Then	
$(m\overline{AB})^2 + (m\overline{AC})^2 = (m\overline{BD})^2 + (m\overline{CD})^2 + 2(m\overline{AD})^2$ $- 2(m\overline{BD})(m\overline{FD}) + 2(m\overline{CD})(m\overline{FD}) \dots\dots(iii)$	Adding (i) and (ii)
Also $m\overline{BD} = m\overline{CD} \dots\dots(iv)$	Given
So	
$(m\overline{AB})^2 + (m\overline{AC})^2 = (m\overline{BD})^2 + (m\overline{BD})^2 + 2(m\overline{AD})^2$ $- \cancel{2(m\overline{BD})(m\overline{FD})} + \cancel{2(m\overline{BD})(m\overline{FD})}$ $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$	Using (iii) and (iv)

Example 1: In $\triangle ABC$, $\angle C$ is obtuse, $\overline{AD} \perp \overline{BC}$ produced, whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$.

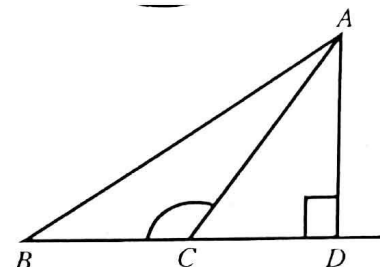
Prove that $(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2(m\overline{BC})(m\overline{BD})$

Given:

In a $\triangle ABC$, $\angle BCA$ is obtuse so that $\angle B$ is acute, $\overline{AD} \perp \overline{BC}$ produced whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$ produced.

To prove: $(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2(m\overline{BC})(m\overline{BD})$

Proof:



Statements	Reasons
In $\angle$ rt $\triangle ABD$ $(m\overline{AB})^2 + (m\overline{AD})^2 = (m\overline{BD})^2$(i)	Pythagoras theorem
In $\angle$ rt $\triangle ACD$ $(m\overline{AC})^2 + (m\overline{AD})^2 = (m\overline{CD})^2$(ii) or $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{BD} - m\overline{BC})^2$ $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{BD})^2 + (m\overline{BC})^2 - 2(m\overline{BC})(m\overline{BD})$(iii) $(m\overline{AC})^2 = (m\overline{AB})^2 + (m\overline{BC})^2 - 2(m\overline{BC})(m\overline{BD})$	Pythagoras theorem $m\overline{BC} = m\overline{CD} + m\overline{BD}$ Using (i) and (iii)

Example 2: In an isosceles $\triangle ABC$, if

$m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$, then prove that $(m\overline{BC})^2 = 2(m\overline{AC})(m\overline{CE})$

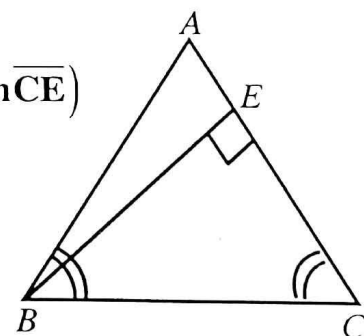
Given: In an isosceles $\triangle ABC$

$m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$

Whereas $\overline{CE}$ is the projection of $\overline{BC}$ on $\overline{AC}$

To prove: $(m\overline{BC})^2 = 2(m\overline{AC})(m\overline{CE})$

Proof:



Statements	Reasons
In an isosceles $\triangle ABC$ with $m\overline{AB} = m\overline{AC}$. If $\angle C$ is acute, then $(m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{BC})^2 - 2(m\overline{AC})(m\overline{CE})$ $(m\overline{AC})^2 = (m\overline{AC})^2 + (m\overline{BC})^2 - 2(m\overline{AC})(m\overline{CE})$ $(m\overline{BC})^2 - 2(m\overline{AC})(m\overline{CE}) = 0$ or $(m\overline{BC})^2 = 2(m\overline{AC})(m\overline{CE})$	By theorem 2 Given $m\overline{AB} = m\overline{AC}$ Cancel $(m\overline{AC})^2$ on both sides