

Exercise 2.5

Q1. Evaluate

$$\begin{aligned}\text{(i)} \quad & i^7 \\ &= i^6 \cdot i \\ &= (i^2)^3 \cdot i \\ &= (-1)^3 \cdot i \\ &= -1 \cdot i \\ &= -i\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad & i^{50} \\ &= (i^2)^{25} \\ &= (-1)^{25} \\ &= -1\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad & i^{12} \\ &= (i^2)^6 \\ &= (-1)^6 \\ &= 1\end{aligned}$$

$$\begin{aligned}\text{(iv)} \quad & (-i)^8 \\ &= i^8 \\ &= (i^2)^4 \\ &= (-1)^4 \\ &= 1\end{aligned}$$

$$\text{(v)} \quad (-i)^5$$

$$\begin{aligned}
 &= -i^5 \\
 &= -(i^4 \cdot i) \\
 &= -((i^2)^2 \cdot i) \\
 &= -((-1)^2 \cdot i) \\
 &\quad -(i) \\
 &= -i
 \end{aligned}$$

(vi) i^{27}

$$\begin{aligned}
 &= i^{26} \cdot i \\
 &= (i^2)^{13} \cdot i \\
 &= (-1)^{13} \cdot i \\
 &= (-1)i \\
 &= -i
 \end{aligned}$$

Q2. Write the conjugate of the following numbers.

(i) $2+3i$

Let $z = 2+3i$

then $\bar{z} = 2-3i$

(ii) $3-5i$

Let $z = 3-5i$

$\bar{z} = 3+5i$

(iii) $-i$

Sol: Let $z = 0-i$

then $\bar{z} = 0+i = i$

(iv) $-3+4i$

Let $z = -3+4i$

then $\bar{z} = -3-4i$

(v) $-4-i$

Let $z = -4-i$

then $\bar{z} = -4+i$

(vi) $i-3$

Let $z = -3+i$

then $\bar{z} = -3-i$

Q3. Write the real and imaginary part of the following number

(i) $1+i$

Let $z = 1+i$

$\text{Re}(z) = 1, \text{Im}(z) = 1$

(ii) $-1+2i$

Let $z = -1+2i$

$\text{Re}(z) = -1, \text{Im}(z) = 2$

(iii) $-3i+2$

Let $z = 2-3i$

$\text{Re}(z) = 2, \text{Im}(z) = -3$

(iv) $-2-2i$

Let $z = -2-2i$

$\text{Re}(z) = -2, \text{Im}(z) = -2$

(v) $-3i$

Let $z = 0-3i$

$\text{Re}(z) = 0, \text{Im}(z) = -3$

(vi) $2+0i$

Let $z = 2+0i$

$\text{Re}(z) = 2, \text{Im}(z) = 0$

Q4. Find the value of x and y if

$$x+iy+1=4-3i$$

Sol: $x+iy+1=4-3i$

$$x+iy=4-1-3i$$

$$x+iy=3-3i$$

Two complex numbers are equal if the real and imaginary parts are equal

So $x=3$ and $y=-3$

Basic Operations on Complex Numbers

(i) Addition:

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in \mathbb{R}$.

The sum of two complex numbers is given by

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

i.e., the sum of two complex numbers is the sum of the corresponding real and the imaginary parts.

$$\text{e.g., } (3 - 8i) + (5 + 2i) = (3 + 5) + (-8 + 2)i = 8 - 6i$$

(ii) Multiplication:

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in \mathbb{R}$.

The products are found as

$$(i) \quad \text{If } k \in \mathbb{R}, kz_1 = k(a + bi) = ka + kbi.$$

(Multiplication of a complex number with a scalar)

$$(ii) \quad z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

(Multiplication of two complex numbers)

The multiplication of any two complex numbers $(a + bi)$ and $(c + di)$ is explained as

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) = a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) && (\text{since } i^2 = -1) \\ &= (ac - bd) + (ad + bc)i && (\text{combining like terms}) \end{aligned}$$

$$\text{e.g., } (2 - 3i)(4 + 5i) = 8 + 10i - 12i - 15i^2 = 23 - 2i. \quad (\text{since } i^2 = -1)$$

(iii) Subtraction:

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in \mathbb{R}$.

The difference between two complex numbers is given by

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

$$\text{e.g., } (-2 + 3i) - (2 + i) = (-2 - 2) + (3 - 1)i = -4 + 2i$$

i.e., the difference of two complex numbers is the difference of the corresponding real and imaginary parts.

iv) Division:

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in \mathbb{R}$.

The division of $a + bi$ by $c + di$ is given by

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

(Multiplying the numerator and denominator by $c - di$, the complex conjugate of $c + di$).

$$= \frac{ac + bci - adi - bdi^2}{c^2 - (di)^2}$$

$$= \frac{ac + bci - adi + bd}{c^2 + d^2}, \text{ since } i^2 = -1$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2} \right) i$$

Example

Separate the real and imaginary parts of $(-1 + \sqrt{-2})^2$

Solution

$$\text{Let } z = -1 + \sqrt{-2}, \text{ then}$$

$$\begin{aligned} z^2 &= (-1 + \sqrt{-2})^2 = (-1 + i\sqrt{2})^2, \text{ changing to } i\text{-form} \\ &= (-1 + i\sqrt{2})(-1 + i\sqrt{2}) = (-1)(-1 + i\sqrt{2}) + i\sqrt{2}(-1 + i\sqrt{2}) \\ &= 1 - i\sqrt{2} - i\sqrt{2} + 2i^2 = -1 - 2\sqrt{2}i \end{aligned}$$

$$\text{Hence } \operatorname{Re}(z^2) = -1 \text{ and } \operatorname{Im}(z^2) = -2\sqrt{2}$$

Example

Express $\frac{1}{1+2i}$ in the standard form $a + bi$.

Solution

$$\text{We have } \frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$$

(multiplying the numerator and denominator by $1-2i$)

$$= \frac{1-2i}{1-(2i)^2} = \frac{1-2i}{1-4i^2}, \text{ (simplifying)}$$

$$= \frac{1-2i}{5}, \quad (\text{since } i^2 = -1)$$

$$= \frac{1}{5} - \frac{2}{5}i, \text{ which is of the form } a + bi$$

Example

Express $\frac{4+5i}{4-5i}$ in the standard form $a + bi$.

Solution

$$\frac{4+5i}{4-5i} = (4+5i) \cdot \frac{1}{4-5i} \times \frac{4+5i}{4+5i}$$

(multiplying and dividing by the conjugate of $(4-5i)$)

$$= \frac{(4+5i)^2}{(4)^2 - (5i)^2} = \frac{16 + 40i + 25i^2}{16 - 25i^2}$$

(simplifying)

$$= \frac{16 + 40i - 25}{16 + 25} \quad (\text{since } i^2 = -1)$$

$$= \frac{-9 + 40i}{41} = \frac{-9}{41} + \frac{40}{41}i$$

Example

Solve $(3 - 4i)(x + yi) = 1 + 0 \cdot i$ for real numbers x and y , where $i = \sqrt{-1}$.

Solution

$$\text{We have } (3 - 4i)(x + yi) = 1 + 0 \cdot i$$

$$\text{or } 3x + 3iy - 4ix - 4i^2y = 1 + 0 \cdot i$$

$$\text{or } 3x + 3iy - 4ix - 4(-1)y = 1 + 0 \cdot i$$

$$\text{or } 3x + 4y + (3y - 4x)i = 1 + 0 \cdot i$$

Equating the real and imaginary parts, we obtain

$$3x + 4y = 1 \quad \text{and} \quad 3y - 4x = 0$$

Solving these two equations

simultaneously, we have $x = \frac{3}{25}$ and

$$y = \frac{4}{25}$$

