

Exercise 4.3

1. Express each of the following surd in the simplest form.

(i) $\sqrt{180}$

$$= \sqrt{2 \times 2 \times 3 \times 3 \times 5}$$

$$= 2 \times 3 \sqrt{5}$$

$$= 6\sqrt{5}$$

(ii) $3\sqrt{162}$

$$= 3\sqrt{2 \times 3 \times 3 \times 3 \times 3}$$

$$= 3(3 \times 3)\sqrt{2}$$

$$= 27\sqrt{2}$$

(iii) $\frac{3}{4}\sqrt[3]{128}$

$$= \frac{3}{4}(128)^{\frac{1}{3}}$$

$$= \frac{3}{4}(2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)^{\frac{1}{3}}$$

$$= \frac{3}{4}(2^3 \times 2^3 \times 2)^{\frac{1}{3}}$$

$$= \frac{3}{4}(2^3)^{\frac{1}{3}} \times (2^3)^{\frac{1}{3}} \times 2^{\frac{1}{3}}$$

$$= \frac{3}{4}(2)(2) \times \sqrt[3]{2}$$

$$= 3\sqrt[3]{2}$$

(iv) $\sqrt[5]{96x^6y^7z^8}$

$$= \sqrt[5]{2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 2^6 y^7 z^8}$$

$$= (2^5 \times 3 \times 2^5 \cdot x \cdot y^5 \cdot y^2 \cdot z^5 \cdot z^3)^{\frac{1}{5}}$$

$$= (2^5)^{\frac{1}{5}} (3)^{\frac{1}{5}} (x^5)^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot (y^5)^{\frac{1}{5}} \cdot (y^2)^{\frac{1}{5}} \cdot (z^5)^{\frac{1}{5}} (z^3)^{\frac{1}{5}}$$

$$= 2 \cdot 3^{\frac{1}{5}} \cdot x \cdot x^{\frac{1}{5}} y \cdot y^{\frac{2}{5}} \cdot z \cdot z^{\frac{3}{5}}$$

$$= 2xyz \cdot 3^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot y^{\frac{2}{5}} \cdot z^{\frac{3}{5}}$$

$$= 2xyz\sqrt[5]{3xy^2z^3}$$

2. Simplify

$$(i) \quad \frac{\sqrt{18}}{\sqrt{3} \cdot \sqrt{2}} = \frac{\sqrt{3 \cdot 3 \cdot 2}}{\sqrt{3} \cdot \sqrt{2}} = \frac{3\cancel{\sqrt{2}}}{\sqrt{3} \cdot \cancel{\sqrt{2}}}$$

$$= \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3}$$

$$(ii) \quad \frac{\sqrt{21} \times \sqrt{9}}{\sqrt{63}} = \frac{\sqrt{3 \times 7} \times \sqrt{3 \times 3}}{\sqrt{3 \times 3 \times 7}} \\ = \frac{\sqrt{3 \times 7 \times 3 \times 3}}{\sqrt{3 \times 3 \times 7}}$$

$$= \frac{\cancel{3}\sqrt{21}}{\cancel{3}\sqrt{7}} = \sqrt{\frac{21}{7}} \\ = \sqrt{3}$$

$$(iii) \quad \sqrt[5]{243x^5y^{10}z^{15}} \\ = (3^5 \cdot x^5 y^{10} z^{15})^{\frac{1}{5}} \\ = (3^5)^{\frac{1}{5}} (x^5)^{\frac{1}{5}} (y^{10})^{\frac{1}{5}} (z^{15})^{\frac{1}{5}} \\ = 3xy^2z^3$$

$$(iv) \quad \frac{4}{5}\sqrt[3]{125} \\ = \frac{4}{5} \left(\cancel{5}^3 \right)^{\frac{1}{3}} \\ = 4$$

$$(v) \quad \sqrt{21} \times \sqrt{7} \times \sqrt{3} \\ = \sqrt{3 \times 7} \times \sqrt{7} \times \sqrt{3} \\ = \sqrt{3 \times 7 \times 7 \times 3} = (3^2 \times 7^2)^{\frac{1}{2}} \\ = (3^2)^{\frac{1}{2}} \times (7^2)^{\frac{1}{2}} \\ = 3 \times 7$$

$$= 21$$

3. Simplify by combining similar terms:

$$(i) \quad \sqrt{45} - 3\sqrt{20} + 4\sqrt{5} \\ = \sqrt{9 \times 5} - 3\sqrt{4 \times 5} + 4\sqrt{5} \\ = 3\sqrt{5} - 6\sqrt{5} + 4\sqrt{5} \\ = (3 - 6 + 4)\sqrt{5} \\ = (-3 + 4)\sqrt{5} \\ = \sqrt{5}$$

$$(ii) \quad 4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300} \\ = 4\sqrt{3 \times 4} + 5\sqrt{3 \times 3 \times 3} - 3\sqrt{3 \times 5 \times 5} \\ + \sqrt{3 \times 2 \times 5 \times 2 \times 5} \\ = 8\sqrt{3} + 15\sqrt{3} - 15\sqrt{3} + 10\sqrt{3} \\ = (8 + 15 - 15 + 10)\sqrt{3} \\ = 18\sqrt{3}$$

$$(iii) \quad \sqrt{3}(2\sqrt{3} + 3\sqrt{3}) \\ = \sqrt{3}((2+3)\sqrt{3}) \\ = \sqrt{3}(5\sqrt{3}) \\ = 5\sqrt{3} \times \sqrt{3} \\ = 5(\sqrt{3 \times 3}) \\ = 5(3) \\ = 15$$

$$(iv) \quad 2(6\sqrt{5} - 3\sqrt{5}) \\ = 2((6-3)\sqrt{5}) \\ = 2(3\sqrt{5}) \\ = 6\sqrt{5}$$

4. Simplify:

$$(i) \quad (3 + \sqrt{3})(3 - \sqrt{3}) \\ = (3)^2 - (\sqrt{3})^2$$

$$\begin{aligned}
&= 9 - 3 \\
&= 6 \\
\text{(ii)} \quad &(\sqrt{5} + \sqrt{3})^2 \\
&= (\sqrt{5})^2 + (\sqrt{3})^2 + 2\sqrt{5}\sqrt{3} \\
&= 5 + 3 + 2\sqrt{15} \\
&= 8 + 2\sqrt{15} \\
\text{(iii)} \quad &(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) \\
&= (\sqrt{5})^2 - (\sqrt{3})^2 \\
&= 5 - 3 \\
&= 2 \\
\text{(iv)} \quad &\left(\sqrt{2} + \frac{1}{\sqrt{3}}\right)\left(\sqrt{2} - \frac{1}{\sqrt{3}}\right) \\
&= (\sqrt{2})^2 - \left(\frac{1}{\sqrt{3}}\right)^2
\end{aligned}$$

$$\begin{aligned}
&= 2 - \frac{1}{3} \\
&= \frac{6-1}{3} = \frac{5}{3} \\
\text{(v)} \quad &(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})(x + y) \\
&\quad (x^2 + y^2) \\
&= \left((\sqrt{x})^2 - (\sqrt{y})^2\right)((x + y)(x^2 + y^2)) \\
&= (x - y)(x + y)(x^2 + y^2) \\
&= (x^2 - y^2)(x^2 + y^2) \\
&= (x^2)^2 - (y^2)^2 \\
&= x^4 - y^4
\end{aligned}$$

Define monomial surd

- (i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2}, \sqrt{3}$ etc.
- (ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.
e.g., $\sqrt{3} + \sqrt{7}$ or $\sqrt{2} + 5$ $\sqrt{11} - 8$ etc.
- (iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

Example

Rationalize the denominator $\frac{58}{7-2\sqrt{5}}$

Solution

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7+2\sqrt{5})$ of $(7-2\sqrt{5})$, i.e.

$$\frac{58}{7-2\sqrt{5}} = \frac{58}{7-2\sqrt{5}} \times \frac{7+2\sqrt{5}}{7+2\sqrt{5}} = \frac{58(7+2\sqrt{5})}{(7)^2 - (2\sqrt{5})^2}$$

$$\begin{aligned}
 &= \frac{58(7+2\sqrt{5})}{49-20}; \text{ (radical is eliminated in the denominator)} \\
 &= \frac{58(7+2\sqrt{5})}{29} = 2(7+2\sqrt{5})
 \end{aligned}$$

Example

Rationalize the denominator $\frac{2}{\sqrt{5}+\sqrt{2}}$

Solution

Multiplying both the numerator and denominator by the conjugate $(\sqrt{5}-\sqrt{2})$ of $(\sqrt{5}+\sqrt{2})$, to get

$$\begin{aligned}
 \frac{2}{\sqrt{5}+\sqrt{2}} &= \frac{2}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{2(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{2(\sqrt{5}-\sqrt{2})}{5-2} \\
 &= \frac{2(\sqrt{5}-\sqrt{2})}{3} = \frac{2(\sqrt{5}-\sqrt{2})}{3}
 \end{aligned}$$

Example

Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution

First we shall rationalize the denominators and then simplify. We have

$$\begin{aligned}
 &\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \\
 = &\frac{6}{2\sqrt{3}-\sqrt{6}} \times \frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} \\
 = &\frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2} \\
 = &\frac{6(2\sqrt{3}+\sqrt{6})}{12-6} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2} \\
 = &\frac{12\sqrt{3}+6\sqrt{6}}{6} + \frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1} - \frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4} \\
 = &2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6} = 0
 \end{aligned}$$

Example

Find rational numbers x and y such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = x + y\sqrt{5}$

Solution

$$\begin{aligned}\frac{4+3\sqrt{5}}{4-3\sqrt{5}} &= \frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} = \frac{(4+3\sqrt{5})^2}{(4)^2 - (3\sqrt{5})^2} \\ &= \frac{16+24\sqrt{5}+45}{16-45} = \frac{61+24\sqrt{5}}{-29}\end{aligned}$$

$$\Rightarrow \frac{-61}{29} - \frac{24}{29}\sqrt{5} = x + y\sqrt{5} \quad (\text{given})$$

Hence, on comparing the two sides, we get

$$x = \frac{-61}{29}, \quad y = \frac{-24}{29}$$

Example

If $x = 3 + \sqrt{8}$, then evaluate

(i) $x + \frac{1}{x}$ and (ii) $x^2 + \frac{1}{x^2}$

Solution

Since $x = 3 + \sqrt{8}$, therefore,

$$\begin{aligned}\frac{1}{x} &= \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2 - (\sqrt{8})^2} \\ &= \frac{3-\sqrt{8}}{9-8} = 3-\sqrt{8}\end{aligned}$$

(i) $x + \frac{1}{x} = 3 + \sqrt{8} + 3 - \sqrt{8} = 6$

(ii) $\left(x + \frac{1}{x}\right)^2 = 36$

or $x^2 + 2x \times \frac{1}{x} + \frac{1}{x^2} = 36$

or $x^2 + \frac{1}{x^2} = 34$

