

FACTORIZATION

Factorization: If a polynomial $p(x)$ can be expressed as $p(x) = g(x) h(x)$, then each of the polynomials $g(x)$ and $h(x)$ is called a **factor of $p(x)$** . The process of finding the factors is called **factorization**.

- (a) **Factorization of the Expression of the type $ka + kb + kc$.**

Example

Factorize $5a - 5b + 5c$

Solution:

$$5a - 5b + 5c = 5(a - b + c)$$

Example

Factorize $5a - 5b - 15c$

Solution:

$$5a - 5b - 15c = 5(a - b - 3c)$$

- (b) **Factorization of the Expression of the type $ac + ad + bc + bd$**

We can write $ac + ad + bc + bd$ as

$$(ac + ad) + (bc + db)$$

$$= a(c + d) + b(c + d)$$

$$= (a + b)(c + d)$$

Example

Factorize $3x - 3a + xy - ay$

Solution:

Regrouping the terms of given polynomial

$$3x + xy - 3a - ay = x(3 + y) - a(3 + y)$$

$$= (3 + y)(x - a)$$

- (d) **Factorization of the Expression of the type $a^2 - b^2$.**

Example

Factorize

Example

Factorize $pqr + qr^2 - pr^2 - r^3$

Solution:

The given expression $= r(pq + qr - pr - r^2)$

$$= r[pq + qr - pr - r^2]$$

$$= r[q(p + r) - r(p + r)]$$

$$= r(p + r)(q - r)$$

- (c) **Factorization of the Expression of the type $a^2 \pm 2ab + b^2$.**

We know that

$$(i) a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$$

$$(ii) a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b)$$

Example

Factorization $25x^2 + 16 + 40x$.

Solution:

$$25x^2 + 40x + 16 = (5x)^2 + 2(5x)(4) + (4)^2$$

$$= (5x + 4)^2$$

$$= (5x + 4)(5x + 4)$$

Example

Factorize $12x^2 - 36x + 27$

Solution:

$$12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$$

$$= 3[(2x)^2 - 2(2x)(3) + (3)^2]$$

$$= 3(2x - 3)^2$$

$$= 3(2x - 3)(2x - 3)$$

$$(i) 4x^2 - (2y - z)^2 \quad (ii) 6x^4 - 96$$

Solution

$$\begin{aligned} (i) \quad 4x^2 - (2y - z)^2 &= (2x)^2 - (2y - z)^2 \\ &= [2x - (2y - z)][2x + (2y - z)] \\ &= (2x - 2y + z)(2x + 2y - z) \end{aligned}$$

$$\begin{aligned} (ii) \quad 6x^4 - 96 &= 6(x^4 - 16) \\ &= 6[(x^2)^2 - (4)^2] \\ &= 6(x^2 - 4)(x^2 + 4) \\ &= 6[(x)^2 - (2)^2](x^2 + 4) \\ &= 6(x - 2)(x + 2)(x^2 + 4) \end{aligned}$$

(e) **Factorization of the Expression of the types $a^2 \pm 2ab + b^2 - c^2$.**

We know that

$$a^2 \pm 2ab + b^2 - c^2 = (a \pm b)^2 - (c)^2 = (a \pm b - c)(a \pm b + c)$$

Example

$$\begin{aligned} \text{Factorize} \quad (i) \quad x^2 + 6x + 9 - 4y^2 \\ (ii) \quad 1 + 2ab - a^2 - b^2 \end{aligned}$$

Solution:

$$\begin{aligned} (i) \quad x^2 + 6x + 9 - 4y^2 &= (x + 3)^2 - (2y)^2 \\ &= (x + 3 + 2y)(x + 3 - 2y) \end{aligned}$$

$$\begin{aligned} (ii) \quad 1 + 2ab - a^2 - b^2 &= 1 - (a^2 - 2ab + b^2) \\ &= (1)^2 - (a - b)^2 \\ &= [1 - (a - b)][1 + (a - b)] \\ &= (1 - a + b)(1 + a - b) \end{aligned}$$

Exercise 5.1

Q.1

Factorize

$$\begin{aligned} (i) \quad 2abc - 4abx + 2abd \\ &= 2ab(c - 2x + d) \end{aligned}$$

$$(ii) \quad 9xy - 12x^2y + 18y^2$$

$$\begin{aligned} &= 3y(3x - 4x^2 + 6y) \\ (iii) \quad -3x^2y - 3x + 9xy^2 \\ &= -3x(xy + 1 - 3y^2) \end{aligned}$$

$$(iv) \quad 5ab^2c^3 - 10a^2b^3c + 20a^3bc^2 \\ = 5abc(bc^2 - 2ab^2 + 4a^2c)$$

$$(v) \quad 3x^3y(x-3y) - 7x^2y^2(x-3y) \\ (x-3y)(3x^3y - 7x^2y^2) \\ (x-3y) \cdot x^2y(3x-7y)$$

$$\Rightarrow x^2y(x-3y)(3x-7y)$$

$$(vi) \quad 2xy^3(x^2+5) + 8xy^2(x^2+5) \\ (x^2+5)(2xy^3 + 8xy^2) \\ (x^2+5) 2xy^2(y+4) \\ = 2xy^2(x^2+5)(y+4)$$

Q.2 (i) $5ax - 3ay - 5bx + 3by$
 $= 5ax - 5bx - 3ay + 3by$
 $= 5x(a-b) - 3y(a-b)$
 $= (a-b)(5x-3y)$

(ii) $3xy + 2y - 12x - 8$
 $= 3xy - 12x + 2y - 8$
 $= 3x(y-4) + 2(y-4)$
 $= (y-4)(3x+2)$

(iii) $x^3 + 3xy^2 - 2x^2y - 6y^3$
 $= x^3 - 2x^2y + 3xy^2 - 6y^3$
 $= x^2(x-2y) + 3y^2(x-2y)$
 $= (x-2y)(x^2+3y^2)$

(iv) $(x^2 - y^2)z + (y^2 - z^2)x$
 $= x^2z - y^2z + y^2x - z^2x$
 $= x^2z - z^2x + y^2x - y^2z$
 $= xz(x-z) + y^2(x-z)$
 $= (x-z)(xz + y^2)$

Q.3 (i) $144a^2 + 24a + 1$
 $= (12a)^2 + 2(12a)(1) + (1)^2$

$$= (12a+1)^2 \\ = (12a+1)(12a+1)$$

(ii) $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$
 $= \left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2$
 $= \left(\frac{a}{b} - \frac{b}{a}\right)^2$
 $= \left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{b} - \frac{b}{a}\right)$

(iii) $(x+y)^2 - 14z(x+y) + 49z^2$
 $= (x+y)^2 - 2(x+y)(7z) + (7z)^2$
 $= (x+y-7z)^2$
 $= (x+y-7z)(x+y-7z)$

(iv) $12x^2 - 36x + 27$
 $= 3(4x^2 - 12x + 9)$
 $= 3[(2x)^2 - 2(2x)(3) + (3)^2]$
 $= 3(2x-3)^2$
 $= 3(2x-3)(2x-3)$

Q.4 (i) $3x^2 - 75y^2$
 $= 3(x^2 - 25y^2)$
 $= 3[(x)^2 - (5y)^2]$
 $= 3(x+5y)(x-5y)$

(ii) $x(x-1) - y(y-1)$
 $= x^2 - x - y^2 + y$
 $= x^2 - y^2 - x + y$
 $= (x+y)(x-y) - 1(x-y)$
 $= (x-y)(x+y-1)$

$$\begin{aligned}
 \text{(iii)} \quad & 128am^2 - 242an^2 \\
 &= 2a(64m^2 - 121n^2) \\
 &= 2a[(8m)^2 - (11n)^2] \\
 &= 2a(8m+11n)(8m-11n)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & 3x - 243x^3 \\
 &= 3x(1 - 81x^2) \\
 &= 3x[(1)^2 - (9x)^2] \\
 &= 3x(1+9x)(1-9x)
 \end{aligned}$$

Q.5

$$\begin{aligned}
 \text{(i)} \quad & x^2 - y^2 - 6y - 9 \\
 &= x^2 - (y^2 + 6y + 9) \\
 &= x^2 - [(y)^2 + 2(y)(3) + (3)^2] \\
 &= (x)^2 - (y+3)^2 \\
 &= [(x) + (y+3)][(x) - (y+3)] \\
 &= (x+y+3)(x-y-3)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & x^2 - a^2 + 2a - 1 \\
 &= x^2 - (a^2 - 2a + 1) \\
 &= (x)^2 - (a-1)^2 \\
 &= [(x) + (a-1)][(x) - (a-1)] \\
 &= (x+a-1)(x-a+1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & 4x^2 - y^2 - 2y - 1 \\
 &= 4x^2 - (y^2 + 2y + 1) \\
 &= (2x)^2 - (y+1)^2 \\
 &= [(2x) + (y+1)][(2x) - (y+1)] \\
 &= (2x+y+1)(2x-y-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & x^2 - y^2 - 4x - 2y + 3 \\
 &= x^2 - y^2 - 4x - 2y + 4 - 1
 \end{aligned}$$

$$\begin{aligned}
 &= x^2 - 4x + 4 - y^2 - 2y - 1 \\
 &= (x)^2 - 2(x)(2) + (2)^2 - (y^2 + 2y + 1) \\
 &= (x-2)^2 - (y+1)^2 \\
 &= [(x-2) + (y+1)][(x-2) - (y+1)] \\
 &= (x-2+y+1)(x-2-y-1) \\
 &= (x+y-1)(x-y-3)
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & 25x^2 - 10x + 1 - 36z^2 \\
 &= (5x)^2 - 2(5x)(1) + (1)^2 - (6z)^2 \\
 &= (5x-1)^2 - (6z)^2 \\
 &= [(5x-1) + (6z)][(5x-1) - (6z)] \\
 &= (5x-1+6z)(5x-1-6z) \\
 &= (5x+6z-1)(5x-6z-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & x^2 - y^2 - 4xz + 4z^2 \\
 &= x^2 - 4xz + 4z^2 - y^2 \\
 &= (x)^2 - 2(x)(2z) + (2z)^2 - (y)^2 \\
 &= (x-2z)^2 - (y)^2 \\
 &= [(x-2z) + (y)][(x-2z) - (y)] \\
 &= (x-2z+y)(x-2z-y)
 \end{aligned}$$

(a) Factorization of the Expression of types $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

Factorization of such types of expression is explained in the following examples.

Example

Factorize $81x^4 + 36x^2y^2 + 16y^4$

Solution

$$\begin{aligned}
 & 81x^4 + 36x^2y^2 + 16y^4 \\
 &= (9x^2)^2 + 72x^2y^2 + (4y^2)^2 - 36x^2y^2 \\
 &= (9x^2)^2 + (4y^2)^2 + 2(9x^2)(4y^2) - 36x^2y^2
 \end{aligned}$$

$$\begin{aligned}
 &= (9x^2 + 4y^2)^2 - (6xy)^2 \\
 &= (9x^2 + 4y^2 + 6xy)(9x^2 + 4y^2 - 6xy) \\
 &= (9x^2 + 6xy + 4y^2)(9x^2 - 6xy + 4y^2)
 \end{aligned}$$

Example

Factorize $9x^4 + 36y^4$

Solution:

$$\begin{aligned}
 &9x^4 + 36y^4 \\
 &= 9x^4 + 36y^4 + 36x^2y^2 - 36x^2y^2 \\
 &= (3x^2)^2 + 2(3x^2)(6y^2) + (6y^2)^2 - (6xy)^2 \\
 &= (3x^2 + 6y^2)^2 - (6xy)^2 \\
 &= (3x^2 + 6y^2 + 6xy)(3x^2 + 6y^2 - 6xy) \\
 &= (3x^2 + 6xy + 6y^2)(3x^2 - 6xy + 6y^2)
 \end{aligned}$$

(b) **Factorization of the Expression of the type $x^2 + px + q$.**

Example

Factorize (i) $x^2 - 7x + 12$
(ii) $x^2 + 15x - 36$

Solution:

(i) $x^2 - 7x + 12$

From the factors of 12 the suitable pair of numbers is -3 and -4 since

$$(-3) + (-4) = -7 \text{ and } (-3)(-4) = 12$$

$$\text{Hence } x^2 - 7x + 12 = x^2 - 3x - 4x + 12$$

$$= x(x-3) - 4(x-3)$$

$$= (x-3)(x-4)$$

(ii) $x^2 + 5x - 36$

From the possible factors of 36, the suitable pair is 9 and -4 because

$$9 + (-4) = 5 \text{ and } 9 \times (-4) = -36$$

$$\text{Hence } x^2 + 5x - 36 = x^2 + 9x - 4x - 36$$

$$= x(x+9) - 4(x+9)$$

$$= (x+9)(x-4)$$

(c) **Factorization of the Expression of the type $ax^2 + bx + c$, $a \neq 0$**

Example

Factorize (i) $9x^2 + 21x - 8$

(ii) $2x^2 - 8x - 42$

(iii) $10x^2 - 41xy + 21y^2$

Solution:

(i) $9x^2 + 21x - 8$

In this case, on comparing with

$$ax^2 + bx + c, \quad ac = (9)(-8) = -72.$$

From the possible factors of 72 the suitable pair of numbers (with proper sign) is 24 and -3 whose
Sum = $24 + (-3) = 21$, (the coefficient of x)

$$\text{And their product} = (24)(-3) = -72 = ac$$

$$\text{Hence } 9x^2 + 21x - 8$$

$$= 9x^2 + 24x - 3x - 8$$

$$= 3x(3x+8) - 1(3x+8)$$

$$= (3x+8)(3x-1)$$

(ii) $2x^2 - 8x - 42 = 2(x^2 - 4x - 21)$

Comparing

$$x^2 - 4x - 21 \text{ with } ax^2 + bx + c$$

$$\text{We have } ac = (+1)(-21) = -21$$

From the possible factors of 21 the suitable pair of numbers is -7 and $+3$ whose

$$\text{Sum} = -7 + 3 = -4 \text{ and product} = (-7)(3) = -21$$

$$\text{Hence } x^2 - 4x - 21$$

$$= x^2 + 3x - 7x - 21$$

$$= x(x+3) - 7(x+3)$$

$$= (x+3)(x-7)$$

$$\begin{aligned} \text{Hence } 2x^2 - 8x - 42 &= 2(x^2 - 4x - 21) \\ &= 2(x+3)(x-7) \end{aligned}$$

(iii) $10x^2 - 41xy + 21y^2$

Here $ac = (10)(21) = 210$

Two suitable factors of 210 are -35 and -6 .

Their sum $= -35 - 6 = -41$

And product $= (-35)(-6) = 210$

Hence $10x^2 - 41xy + 21y^2$

$$\begin{aligned} &= 10x^2 - 35xy - 6xy + 21y^2 \\ &= 5x(2x - 7y) - 3y(2x - 7y) \\ &= (2x - 7y)(5x - 3y) \end{aligned}$$

(d) **Factorization of the following types of Expressions.**

$$(ax^2 + b + c)(ax^2 + bx + d) + k$$

$$(x+a)(x+b)(x+c)(x+d) + k$$

$$(x+a)(x+b)(x+c)(x+d) + kx^2$$

Example

Factorize $(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Solution:

$$(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$$

Let $y = x^2 - 4x$. Then

$$\begin{aligned} (y-5)(y-12) - 144 &= y^2 - 17y + 60 - 144 \\ &= y^2 - 17y - 84 \\ &= y^2 - 21y + 4y - 84 \\ &= y(y-21) + 4(y-21) \\ &= (y-21)(y+4) \\ &= (x^2 - 4x - 21)(x^2 - 4x + 4) \quad (\text{Since } y = x^2 - 4x) \\ &= (x^2 - 7x + 3x - 21)[(x)^2 - 2(x)(2) + (2)^2] \\ &= [x(x-7) + 3(x-7)](x-2)^2 \\ &= (x-7)(x+3)(x-2)(x-2) \end{aligned}$$

Example

Factorize

$$(x+1)(x+2)(x+3)(x+4)-120$$

Solution:We observe that $1+4=2+3$.

It suggests that we rewrite the given expression as

$$[(x+1)(x+4)][(x+2)(x+3)]-120$$

$$(x^2+5x+4)(x^2+5x+6)-120$$

Let $x^2+5x=y$, then

$$\text{We get } (y+4)(y+6)-120$$

$$=y^2+10y+24-120$$

$$=y^2+10y-96$$

$$=y^2+16y-6y-96$$

$$=y(y+16)-6(y+16)$$

$$=(y+16)(y-6)$$

$$=(x^2+5x+16)(x^2+5x-6) \text{ (since } y=x^2+5x)$$

$$=(x^2+5x+16)[x^2+6x-x-6]$$

$$=(x^2+5x+16)[(x+6)-1(x+6)]$$

$$=(x^2+5x+16)(x+6)(x-1)$$

Example

$$\text{Factorize } (x^2-5x+6)(x^2+5x+6)-2x^2$$

Solution:

$$(x^2-5x+6)(x^2+5x+6)-2x^2$$

$$=[x^2-3x-2x+6][x^2+3x+2x+6]-2x^2$$

$$=[x(x-3)-2(x-3)][x(x+3)+2(x+3)]-2x^2$$

$$=[(x-3)(x-2)][(x+3)(x+2)]-2x^2$$

$$=[(x-2)(x+2)][(x-3)(x+3)]-2x^2$$

$$=(x^2-4)(x^2-9)-2x^2$$

$$=x^4-13x^2+36-2x^2$$

$$=x^4-15x^2+36$$

$$=x^4-12x^2-3x^2+36$$

$$=x^2(x^2-12)-3(x^2-12)$$

$$=(x^2-12)(x^2-3)$$

$$=[(x)^2-(2\sqrt{3})^2][(x)^2-(\sqrt{3})^2]$$

$$=(x-2\sqrt{3})(x+2\sqrt{3})(x-\sqrt{3})(x+\sqrt{3})$$

(e) Factorization of Expressions of the following Types

$$a^3+3a^2b+3ab^2+b^3$$

$$a^3-3a^2b+3ab^2-b^3$$

Example:

$$\text{Factorize } x^3-8y^3-6x^2y+12xy^2$$

Solution:

$$x^3-8y^3-6x^2y+12xy^2$$

$$=(x)^3-(2y)^3-3(x)^2(2y)+3(x)(2y)^2$$

$$=(x)^3-3(x)^2(2y)+3(x)(2y)^2-(2y)^3$$

$$=(x-2y)^3$$

$$=(x-2y)(x-2y)(x-2y)$$

(d) Factorization of Expressions of the following types $a^3 \pm b^3$

We recall the formulas,

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Example

$$\text{Factorize } 27x^3+64y^3$$

Solution:

$$27x^3+64y^3=(3x)^3+(4y)^3$$

$$=(3x+4y)[(3x)^2-(3x)(4y)+(4y)^2]$$

$$=(3x+4y)(9x^2-12xy+16y^2)$$

Example

Factorize $1-125x^3$

Solution

$$1-25x^3 = (1)^3 - (5x)^3$$

$$= (1-5x) \left[(1)^2 + (1)(5x) + (5x)^2 \right]$$

$$= (1-5x)(1+5x+25x^2)$$