

Exercise 5.2

Q.1 Factorize

$$(i) \quad x^4 + \frac{1}{x^4} - 3$$

$$= x^4 + \frac{1}{x^4} - 2 - 1$$

$$= (x^2)^2 + \left(\frac{1}{x^2}\right)^2 - 2(x^2)\left(\frac{1}{x^2}\right) - 1$$

$$= \left(x^2 - \frac{1}{x^2}\right)^2 - (1)^2$$

$$= \left(x^2 - \frac{1}{x^2} + 1\right)\left(x^2 - \frac{1}{x^2} - 1\right)$$

$$(ii) \quad 3x^4 + 12y^4$$

$$= 3(x^4 + 4y^4)$$

$$= 3[(x^2)^2 + (2y^2)^2 + 2(x^2)(2y^2) - 4x^2y^2]$$

$$= 3[(x^2 + 2y^2)^2 - (2xy)^2]$$

$$= 3(x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$$

$$= 3(x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

$$(iii) \quad a^4 + 3a^2b^2 + 4b^4$$

$$a^4 + 4a^2b^2 + 4b^4 - a^2b^2$$

$$= (a^2)^2 + 2(a^2)(2b^2) + (2b^2)^2 - a^2b^2$$

$$= (a^2 + 2b^2)^2 - (ab)^2$$

$$= (a^2 + 2b^2 + ab)(a^2 + 2b^2 - ab)$$

$$= (a^2 + ab + 2b^2)(a^2 - ab + 2b^2)$$

$$(iv) \quad 4x^4 + 81$$

$$= (2x^2)^2 + (9)^2 + 2(2x^2)(9) - 36x^2$$

$$= (2x^2 + 9)^2 - (6x)^2$$

$$= (2x^2 + 9 + 6x)(2x^2 + 9 - 6x)$$

$$= (2x^2 + 6x + 9)(2x^2 - 6x + 9)$$

$$(v) \quad x^4 + x^2 + 25$$

$$= (x^2)^2 + 2(x^2)(5) + (5)^2 - 9x^2$$

$$= (x^2 + 5)^2 - (3x)^2$$

$$= (x^2 + 5 + 3x)(x^2 + 5 - 3x)$$

$$= (x^2 + 3x + 5)(x^2 - 3x + 5)$$

$$(vi) \quad x^4 + 4x^2 + 16$$

$$= (x^2)^2 + 2(x^2)(4) + (4)^2 - 4x^2$$

$$= (x^2 + 4)^2 - (2x)^2$$

$$= (x^2 + 4 + 2x)(x^2 + 4 - 2x)$$

$$= (x^2 + 2x + 4)(x^2 - 2x + 4)$$

Q.2

$$(i) \quad x^2 + 14x + 48$$

$$= x^2 + 6x + 8x + 48$$

$$= x(x + 6) + 8(x + 6)$$

$$= (x + 6)(x + 8)$$

$$(ii) \quad x^2 - 21x + 108$$

$$= x^2 - 9x - 12x + 108$$

$$= x(x - 9) - 12(x - 9)$$

$$\begin{aligned}
 &= (x-9)(x-12) \\
 \text{(iii)} \quad &x^2 - 11x - 42 \\
 &= x^2 + 3x - 14x - 42 \\
 &= x(x+3) - 14(x+3) \\
 &= (x+3)(x-14)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad &x^2 + x - 132 \\
 &= x^2 + 12x - 11x - 132 \\
 &= x(x+12) - 11(x+12) \\
 &= (x+12)(x-11)
 \end{aligned}$$

$$\begin{aligned}
 \text{Q.3 (i)} \quad &4x^2 + 12x + 5 \\
 &= 4x^2 + 2x + 10x + 5 \\
 &= 2x(2x+1) + 5(2x+1) \\
 &= (2x+1)(2x+5)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad &30x^2 + 7x - 15 \\
 &= 30x^2 + 25x - 18x - 15 \\
 &= 5x(6x+5) - 3(6x+5) \\
 &= (6x+5)(5x-3)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad &24x^2 - 65x + 21 \\
 &= 24x^2 - 56x - 9x + 21 \\
 &= 8x(3x-7) - 3(3x-7) \\
 &= (3x-7)(8x-3)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad &5x^2 - 16x - 21 \\
 &= 5x^2 + 5x - 21x - 21 \\
 &= 5x(x+1) - 21(x+1) \\
 &= (x+1)(5x-21)
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad &4x^2 - 17xy + 4y^2 \\
 &= 4x^2 - 16xy - xy + 4y^2 \\
 &= 4x(x-4y) - y(x-4y) \\
 &= (x-4y)(4x-y)
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad &3x^2 - 38xy - 13y^2 \\
 &= 3x^2 - 39xy + xy - 13y^2 \\
 &= 3x(x-13y) + y(x-13y) \\
 &= (x-13y)(3x+y)
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad &5x^2 + 33xy - 14y^2 \\
 &= 5x^2 + 35xy - 2xy - 14y^2 \\
 &= 5x(x+7y) - 2y(x+7y) \\
 &= (x+7y)(5x-2y)
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad &\left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4 \\
 &= \left(5x - \frac{1}{x}\right)^2 + 2\left(5x - \frac{1}{x}\right)(2) + (2)^2 \\
 &= \left(5x - \frac{1}{x} + 2\right)^2 \\
 &= \left(5x - \frac{1}{x} + 2\right)\left(5x - \frac{1}{x} + 2\right)
 \end{aligned}$$

$$\text{Q.4 (i)} \quad (x^2 + 5x + 4)(x^2 + 5x + 6) - 3$$

$$\text{Let } x^2 + 5x = y$$

then

$$\begin{aligned}
 &(x^2 + 5x + 4)(x^2 + 5x + 6) - 3 \\
 &= (y+4)(y+6) - 3 \\
 &= y^2 + 4y + 6y + 24 - 3 \\
 &= y^2 + 10y + 21 \\
 &= y^2 + 3y + 7y + 21 \\
 &= y(y+3) + 7(y+3) \\
 &= (y+3)(y+7)
 \end{aligned}$$

Putting value of y

$$= (x^2 + 5x + 3)(x^2 + 5x + 7)$$

$$\text{(ii)} \quad (x^2 - 4x)(x^2 - 4x - 1) - 20$$

Let $x^2 - 4x = y$

then

$$(x^2 - 4x)(x^2 - 4x - 1) - 20$$

$$= y(y - 1) - 20$$

$$= y^2 - y - 20$$

$$= y^2 + 4y - 5y - 20$$

$$= y(y + 4) - 5(y + 4)$$

$$= (y + 4)(y - 5)$$

Putting value of y

$$= (x^2 - 4x + 4)(x^2 - 4x - 5)$$

$$= [(x)^2 - 2(x)(2) + (2)^2][x^2 + x - 5x - 5]$$

$$= (x - 2)^2 [x(x + 1) - 5(x + 1)]$$

$$= (x - 2)^2 (x + 1)(x - 5)$$

(iii) $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

$$= [(x + 2)(x + 5)][(x + 3)(x + 4)] - 15$$

$$= (x^2 + 2x + 5x + 10)(x^2 + 3x + 4x + 12) - 15$$

$$= (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$$

Let $x^2 + 7x = y$

$$= (y + 10)(y + 12) - 15$$

$$= y^2 + 10y + 12y + 120 - 15$$

$$= y^2 + 22y + 105$$

$$= y^2 + 7y + 15y + 105$$

$$= y(y + 7) + 15(y + 7)$$

$$= (y + 7)(y + 15)$$

Putting value of ' y '

$$(x^2 + 7x + 7)(x^2 + 7x + 15)$$

(iv) $(x + 4)(x - 5)(x + 6)(x - 7) - 504$

$$= (x^2 + 4x - 5x - 20)(x^2 + 6x - 7x - 42) - 504$$

$$= (x^2 - x - 20)(x^2 - x - 42) - 504$$

Let $x^2 - x = y$

$$= (y - 20)(y - 42) - 504$$

$$= y^2 - 20y - 42y + 840 - 504$$

$$= y^2 - 62y + 336$$

$$= y^2 - 6y - 56y + 336$$

$$= y(y - 6) - 56(y - 6)$$

$$= (y - 6)(y - 56)$$

Putting value of ' y '

$$= (x^2 - x - 6)(x^2 - x - 56)$$

$$= (x^2 + 2x - 3x - 6)(x^2 + 7x - 8x - 56)$$

$$= [x(x + 2) - 3(x + 2)][x(x + 7) - 8(x + 7)]$$

$$= (x + 2)(x - 3)(x + 7)(x - 8)$$

(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

$$= (x + 1)(x + 6)(x + 2)(x + 3) - 3x^2$$

$$= (x^2 + x + 6x + 6)(x^2 + 2x + 3x + 6) - 3x^2$$

$$= (x^2 + 6 + 7x)(x^2 + 6 + 5x) - 3x^2$$

$$= \frac{x^2}{x^2} [(x^2 + 6 + 7x)(x^2 + 6 + 5x) - 3x^2]$$

$$= x^2 \left[\frac{(x^2 + 6 + 7x)(x^2 + 6 + 5x)}{x^2} - \frac{3x^2}{x^2} \right]$$

$$= x^2 \left[\left(x + \frac{6}{x} + 7 \right) \left(x + \frac{6}{x} + 5 \right) - 3 \right]$$

Let $x + \frac{6}{x} = y$

$$= x^2 [(y + 7)(y + 5) - 3]$$

$$= x^2 (y^2 + 7y + 5y + 35 - 3)$$

$$= x^2 (y^2 + 12y + 32)$$

$$= x^2 (y^2 + 4y + 8y + 32)$$

$$= x^2 [y(y + 4) + 8(y + 4)]$$

$$= x^2 (y + 4)(y + 8)$$

Putting value of y

$$\begin{aligned}
 &= x^2 \left(x + \frac{6}{x} + 4 \right) \left(x + \frac{6}{x} + 8 \right) \\
 &= x^2 \left(\frac{x^2 + 4x + 6}{x} \right) \left(\frac{x^2 + 8x + 6}{x} \right) \\
 &= (x^2 + 4x + 6)(x^2 + 8x + 6) \\
 &= (x^2 + 4x + 6)(x^2 + 8x + 6)
 \end{aligned}$$

Q.5

$$\begin{aligned}
 \text{(i)} \quad &x^3 + 48x - 12x^2 - 64 \\
 &= x^3 - 12x^2 + 48x - 64 \\
 &= (x)^3 - 3(x^2)(4) + 3(x)(4)^2 - (4)^3 \\
 &= (x - 4)^3 \\
 &= (x - 4)(x - 4)(x - 4) \\
 \text{(ii)} \quad &8x^3 + 60x^2 + 150x + 125 \\
 &= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3 \\
 &= (2x + 5)^3 \\
 &= (2x + 5)(2x + 5)(2x + 5) \\
 \text{(iii)} \quad &x^3 - 18x^2 + 108x - 216 \\
 &= (x)^3 - 3(x)^2(6) + 3(x)(6)^2 - (6)^3 \\
 &= (x - 6)^3 \\
 &= (x - 6)(x - 6)(x - 6) \\
 \text{(iv)} \quad &8x^3 - 125y^3 - 60x^2y + 150xy^2 \\
 &= 8x^3 - 60x^2y + 150xy^2 - 125y^3 \\
 &= (2x)^3 - 3(2x)^2(5y) + 3(2x)(5y)^2 - (5y)^3 \\
 &= (2x - 5y)^3 \\
 &= (2x - 5y)(2x - 5y)(2x - 5y)
 \end{aligned}$$

Q.6

$$\begin{aligned}
 \text{(i)} \quad &27 + 8x^3 \\
 &= (3)^3 + (2x)^3 \\
 &= (3 + 2x) \left[(3)^2 - (3)(2x) + (2x)^2 \right] \\
 &= (3 + 2x)(9 - 6x + 4x^2)
 \end{aligned}$$

$$\text{or} \quad = (2x + 3)(4x^2 - 6x + 9)$$

$$\begin{aligned}
 \text{(ii)} \quad &125x^3 - 216y^3 \\
 &= (5x)^3 - (6y)^3 \\
 &= (5x - 6y) \left[(5x)^2 + (5x)(6y) + (6y)^2 \right] \\
 &= (5x - 6y)(25x^2 + 30xy + 36y^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad &64x^3 + 27y^3 \\
 &= (4x)^3 + (3y)^3 \\
 &= (4x + 3y) \left[(4x)^2 - (4x)(3y) + (3y)^2 \right] \\
 &= (4x + 3y)(16x^2 - 12xy + 9y^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad &8x^3 + 125y^3 \\
 &= (2x)^3 + (5y)^3 \\
 &= (2x + 5y) \left[(2x)^2 - (2x)(5y) + (5y)^2 \right] \\
 &= (2x + 5y)(4x^2 - 10xy + 25y^2)
 \end{aligned}$$

Remainder Theorem

If a polynomial $p(x)$ is divided by a linear divisor $(x - a)$, then the remainder is $p(a)$.

Proof

Let $q(x)$ be the quotient obtained after dividing $p(x)$ by $(x - a)$. But the divisor $(x - a)$ is linear. So the remainder must be of degree zero i.e., a non-zero constant, say R . Consequently, by division Algorithm we may write.

$$p(x) = (x - a)q(x) + R$$

This is an identity in x and so is true for all real numbers x . In particular, it is true for $x = a$. Therefore,

$$p(a) = (a - a)q(a) + R = 0 + R = R$$

i.e., $p(a) =$ the remainder.

Hence the theorem.

Note: Similarly, if the divisor is $(ax - b)$, we have

$$p(x) = (ax - b)q(x) + R$$

Substituting $x = \frac{b}{a}$ so that $ax - b = 0$,

we obtain

$$p\left(\frac{b}{a}\right) = 0. \quad q\left(\frac{b}{a}\right) + R = 0 + R = R$$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

To find remainder (without dividing) when a polynomial is divided by a

Linear Polynomial

Example

Find the remainder when

$9x^2 - 6x + 2$ is divided by

- | | | | |
|-------|----------|------|---------|
| (i) | $x - 3$ | (ii) | $x + 3$ |
| (iii) | $3x + 1$ | (iv) | x |

Solution:

Let $p(x) = 9x^2 - 6x + 2$

- (i) When $p(x)$ is divided by $x - 3$, by Remainder Theorem, the remainder is:

$$R = p(3) = 9(3)^2 - 6(3) + 2 = 65$$

$$= 9(9) - 18 + 2$$

$$P(3) = 81 - 18$$

$$= 65$$

- (ii) When $p(x)$ is divided by $x + 3 = x - (-3)$, the remainder is

$$R = p(-3) = 9(-3)^2 - 6(-3) + 2$$

$$= 9(9) + 18 + 2$$

$$= 81 + 20 = 101$$

- (iii) When $p(x)$ is divided by $3x + 1$, the remainder is

$$R = p\left(-\frac{1}{3}\right) = 9\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 2 = 5$$

- (iv) When $p(x)$ is divided by x , the remainder is

$$R = p(0) = 9(0)^2 - 6(0) + 2 = 2$$

Example

Find the value of k is the

expression $x^3 + kx^2 + 3x - 4$ leaves a remainder of -2 when divided by $x + 2$.

Solution:

Let $p(x) = x^3 + kx^2 + 3x - 4$.

By the remainder Theorem, when $p(x)$ is divided by $x + 2 = x - (-2)$, the remainder is:

$$p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4$$

$$= -8 + 4k - 6 - 4$$

$$= 4k - 18$$

By the given condition, we have

$$p(-2) = -2 \Rightarrow 4k - 18 = -2$$

$$\Rightarrow k = 4$$

5.2.3 Zero of a polynomial

If a specific number $x = a$ is substituted for a variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.

Factor Theorem

The polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Proof:

Let $q(x)$ be the quotient and R the remainder when a polynomial $p(x)$ is divided by $(x - a)$. Then by division Algorithm,

$$p(x) = (x - a)q(x) + R$$

By the Remainder Theorem, $R = p(a)$.

Hence $p(x) = (x-a)q(x) + p(a)$

- (i) Now if $p(a) = 0$, then

$$p(x) = (x-a)q(x)$$

i.e., $(x-a)$ is a factor of $p(x)$.

- (ii) Conversely, if $(x-a)$ is a factor of $p(x)$, then the remainder upon dividing $p(x)$ by $(x-a)$ must be zero i.e., $p(a) = 0$.

Example

Determine if $(x-2)$ is a factor of $x^3 - 4x^2 + 3x + 2$.

Solution:

Let

$$p(x) = x^3 - 4x^2 + 3x + 2$$

Then the remainder for $(x-2)$ is:

$$\begin{aligned} p(2) &= (2)^3 - 4(2)^2 + 3(2) + 2 \\ &= 8 - 16 + 6 + 2 = 0 \end{aligned}$$

Hence by Factor Theorem, $(x-2)$ is a factor of the polynomial $p(x)$.

Example

Find a polynomial $p(x)$ of degree 3 that has 2, -1, and 3 as zeros (i.e., roots).

Solution:

Since $x = 2, -1, 3$ are roots of $p(x) = 0$.

So by Factor theorem $(x-2), (x+1)$ and $(x-3)$ are the factors of $p(x)$.

$$\text{Thus } p(x) = a(x-2)(x+1)(x-3)$$

Where any non-zero value can be assigned to a .

Taking $a = 1$, we get

$$\begin{aligned} p(x) &= (x-2)(x+1)(x-3) \\ &= x^3 - 4x^2 + x + 6 \end{aligned}$$

as the required polynomial.