

Exercise 5.3

Q.1 Use the remainder theorem to find the remainder, when.

(i) $3x^3 - 10x^2 + 13x - 6$ is divided by $(x - 2)$

Sol:

Let $P(x) = 3x^3 - 10x^2 + 13x - 6$

When $P(x)$ is divided by $x - 2$ by remainder theorem, the remainder is:

$$\begin{aligned} R &= P(2) = 3(2)^3 - 10(2)^2 + 13(2) - 6 \\ &= 3(8) - 10(4) + 26 - 6 \\ &= 24 - 40 + 26 - 6 \\ &= 50 - 46 \\ &= 4 \end{aligned}$$

(ii) $4x^3 - 4x + 3$ is divided by $(2x - 1)$

Sol:

Let $P(x) = 4x^3 - 4x + 3$ when $P(x)$ is divided by $2x - 1$ by remainder theorem, the remainder is

$$\begin{aligned} R &= P\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right) + 3 \\ &= 4\left(\frac{1}{8}\right) - 2 + 3 \\ &= \frac{1}{2} + 1 \\ &= \frac{1+2}{2} \end{aligned}$$

$$R = \frac{3}{2}$$

- (iii) $6x^4 + 2x^3 - x + 2$ is divided by $(x + 2)$

Sol:

Let $P(x) = 6x^4 + 2x^3 - x + 2$ when $P(x)$ is divided by $x + 2$ by remainder theorem, the remainder is

$$\begin{aligned} R = P(-2) &= 6(-2)^4 + 2(-2)^3 - (-2) + 2 \\ &= 6(16) + 2(-8) + 2 + 2 \\ &= 96 - 16 + 4 \\ &= 80 + 4 \\ R &= 84 \end{aligned}$$

- (iv) $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $2x + 1$

Sol:

Let $p(x) = (2x - 1)^3 + 6(3 + 4x)^2 - 10$ when $P(x)$ is divided by $2x + 1$ by remainder theorem, then remainder is

$$\begin{aligned} R = p\left(-\frac{1}{2}\right) &= \left[2\left(-\frac{1}{2}\right) - 1\right]^3 + 6\left[3 + 4\left(-\frac{1}{2}\right)\right]^2 - 10 \\ &= (-1 - 1)^3 + 6(3 - 2)^2 - 10 \\ &= (-2)^3 + 6(1)^2 - 10 \\ &= -8 + 6 - 10 \\ &= -12 \end{aligned}$$

- (v) $x^3 - 3x^2 + 4x - 14$ is divided by $x + 2$

Sol:

Let $P(x) = x^3 - 3x^2 + 4x - 14$ when $P(x)$ is divided by $x + 2$ by remainder theorem, then remainder is

$$\begin{aligned} R = P(-2) &= (-2)^3 - 3(-2)^2 + 4(-2) - 14 \\ &= -8 - 3(4) - 8 - 14 \\ &= -8 - 12 - 8 - 14 \\ &= -42 \end{aligned}$$

Q.2.

- (i) If $(x+2)$ is a factor of $3x^2 - 4kx - 4k^2$, then find the value(s) of k .

Sol:

$$\text{Let } P(x) = 3x^2 - 4kx - 4k^2$$

As given that $x + 2$ is a factor of $P(x)$, so

$$R = 0$$

$$\text{i.e. } P(-2) = 0$$

$$\text{So } 3(-2)^2 - 4k(-2) - 4k^2 = 0$$

$$12 + 8k - 4k^2 = 0$$

Dividing by 4

$$3 + 2k - k^2 = 0$$

$$3 + 3k - k - k^2 = 0$$

$$3(1+k) - k(1+k) = 0$$

$$(1+k)(3-k) = 0$$

$$\Rightarrow 1+k=0 \text{ or } 3-k=0$$

$$\Rightarrow k=-1 \text{ or } k=3$$

- (ii) If $(x-1)$ is factor of $x^3 - kx^2 + 11x - 6$ then find the value of k .

Sol:

$$P(x) = x^3 - kx^2 + 11x - 6$$

As given that $x - 1$ is a factor of $P(x)$, so

$$R = 0$$

$$P(1) = 0$$

$$(1)^3 - k(1)^2 + 11(1) - 6 = 0$$

$$1 - k + 11 - 6 = 0$$

$$6 - k = 0$$

$$\Rightarrow k = 6$$

Q.3 Without actual long division determine whether

- (i) $(x - 2)$ and $(x - 3)$ are factors of P
 $P(x) = x^3 - 12x^2 + 44x - 48$

Sol:

$$P(x) = x^3 - 12x^2 + 44x - 48$$

Taking $x - 2$

$$R = P(2)$$

$$= (2)^3 - 12(2)^2 + 44(2) - 48$$

$$= 8 - 12(4) + 88 - 48$$

$$= 8 - 48 + 88 - 48$$

$$= 0$$

As the remainder is zero, so $(x - 2)$ is a factor of $P(x)$

$$\text{Now } P(x) = x^3 - 12x^2 + 44x - 48$$

Taking $x - 3$

$$R = P(3)$$

$$= (3)^3 - 12(3)^2 + 44(3) - 48$$

$$= 27 - 12(9) + 132 - 48$$

$$= 27 - 108 + 132 - 48$$

$$= 3 \neq 0$$

As the remainder is not equal to zero, so $(x - 3)$ is not a factor of $P(x)$.

(ii) $(x - 2)$, $(x + 3)$ and $(x - 4)$ are factors of $q(x) = x^3 + 2x^2 - 5x - 6$

Sol:

$$q(x) = x^3 + 2x^2 - 5x - 6$$

Taking $x - 2$

$$R = q(2) = (2)^3 + 2(2)^2 - 5(2) - 6$$

$$= 8 + 2(4) - 10 - 6$$

$$R = 0$$

As the remainder is zero
so $(x - 2)$ is a factor of $P(x)$

$$\text{Now } q(x) = x^3 + 2x^2 - 5x - 6$$

Taking $x + 3$

$$R = q(-3)$$

$$= (-3)^3 + 2(-3)^2 - 5(-3) - 6$$

$$= -27 + 2(9) + 15 - 6$$

$$= -27 + 18 + 15 - 6$$

$$= 0$$

As the remainder is zero, so $(x + 3)$ is a factor of $P(x)$

$$\text{Now } q(x) = x^3 + 2x^2 - 5x - 6$$

Taking $x - 4$

$$R = q(4)$$

$$= (4)^3 + 2(4)^2 - 5(4) - 6$$

$$= 64 + 2(16) - 20 - 6$$

$$= 64 + 32 - 20 - 6$$

$$= 70 \neq 0$$

As remainder is not equal to zero, so $x - 4$ is not a factor of $P(x)$

Q.4 For what value of m is the polynomial $P(x) = 4x^3 - 7x^2 + 6x - 3m$ exactly divisible by $x + 2$?

Sol:

$$m = ?$$

$$P(x) = 4x^3 - 7x^2 + 6x - 3m$$

Taking $x + 2$

As $p(x)$ is exactly divisible by $(x + 2)$, so

$$R = 0$$

$$P(-2) = 0$$

$$4(-2)^3 - 7(-2)^2 + 6(-2) - 3m = 0$$

$$4(-8) - 7(4) - 12 - 3m = 0$$

$$-32 - 28 - 12 - 3m = 0$$

$$-72 - 3m = 0$$

$$-3m = +72$$

$$m = \frac{72}{-3}$$

$$m = -24$$

Q.5 Determine the value of k if

$$P(x) = kx^3 + 4x^2 + 3x - 4 \text{ and}$$

$q(x) = x^3 - 4x + k$. Leaves the same remainder when divided by $x - 3$.

Sol:

$$K = ?$$

When $p(x)$ is divided by $(x-3)$ by remainder theorem then remainder is

$$R_1 = P(3)$$

$$= k(3)^3 + 4(3)^2 + 3(3) - 4$$

$$= 27k + 36 + 9 - 4$$

$$= 27k + 41$$

When $q(x)$ is divided by $(x-3)$ by remainder theorem then remainder is

$$R_2 = q(3)$$

$$q(x) = x^3 - 4x + k$$

$$= (3)^3 - 4(3) + k$$

$$= 27 - 12 + k$$

$$= 15 + k$$

As given that when $P(x)$ and $q(x)$ are divided by $x - 3$, then remainder is same, so

$$R_1 = R_2$$

$$27k + 41 = 15 + k$$

$$27k - k = 15 - 41$$

$$26k = -26$$

$$k = \frac{-26}{26}$$

$$\boxed{k = -1}$$

Q.6

The remainder of dividing the polynomial

$$P(x) = x^3 + ax^2 + 7 \text{ by } (x + 1) \text{ is } 2b.$$

calculate the value of 'a' and 'b' if this expression leaves a remainder of $(b + 5)$ on being divided by $(x - 2)$

Sol:

$$P(x) = x^3 + ax^2 + 7$$

The remainder by dividing

$P(x)$ by $x + 1$ is $2b$, so

$$P(-1) = 2b$$

$$(-1)^3 + a(-1)^2 + 7 = 2b$$

$$-1 + a + 7 = 2b$$

$$a + 6 = 2b$$

$$a - 2b = -6 \dots (i)$$

Taking $x - 2$

The remainder by dividing

$P(x)$ by $(x - 2)$ is $(b + 5)$, so

$$P(2) = b + 5$$

$$(2)^3 + a(2)^2 + 7 = b + 5$$

$$8 + 4a + 7 = b + 5$$

$$4a + 15 = b + 5$$

$$4a - b = 5 - 15$$

$$4a - b = -10 \dots (ii)$$

Multiplying (ii) by 2

$$8a - 2b = -20 \dots (iii)$$

By Subtracting, (iii) from (i)

$$a - 2b = -6$$

$$8a - 2b = -20$$

$$\frac{-7a}{-7a} = \frac{14}{-7a}$$

$$a = -\frac{14}{7} = -2$$

Putting (1)

$$a - 2b = -6$$

$$-2 - 2b = -6$$

$$-2b = -6 + 2$$

$$-2b = -4$$

$$b = 2$$

Q.7 The polynomial

$x^3 + \ell x^2 + mx + 24$ has a factor $(x + 4)$ and it leaves a remainder of 36 when divided by $(x - 2)$. Find the value of ℓ and m .

Sol:

$$\text{Let } P(x) = x^3 + \ell x^2 + mx + 24$$

As $(x + 4)$ is a factor of $P(x)$,

So remainder will be zero. i.e

$$R = P(-4) = 0$$

$$P(-4) = 0$$

$$(-4)^3 + \ell(-4)^2 + m(-4) + 24 = 0$$

$$-64 + 16\ell - 4m + 24 = 0$$

$$16\ell - 4m - 40 = 0$$

$$16\ell - 4m = 40$$

Dividing by 4

$$4\ell - m = 10 \dots (i)$$

Now as given that $P(x)$ is divided by $(x - 2)$ leaves a remainder 36, so

$$R = 36$$

$$\text{i.e. } P(2) = 36$$

$$(2)^3 + \ell(2)^2 + m(2) + 24 = 36$$

$$8 + 4\ell + 2m + 24 = 36$$

$$4\ell + 2m + 32 = 36$$

$$4\ell + 2m = 36 - 32$$

$$4\ell + 2m = 4$$

Dividing by 2

$$2\ell + m = 2 \dots (ii)$$

Adding (i) and (ii)

$$4\ell - m = 10$$

$$2\ell + m = 2$$

$$\hline 6\ell = 12$$

$$\ell = \frac{12}{6}$$

$$\ell = 2$$

Putting value of ' ℓ ' in (ii)

$$2\ell + m = 2$$

$$2(2) + m = 2$$

$$m = 2 - 4$$

$$m = -2$$

Q.8. The Expression $\ell x^3 + mx^2 - 4$ leaves remainder of -3 and 12 when divided by $(x - 1)$ and $(x + 2)$ respectively. Calculate the values of ℓ and m .

Sol:

$$\text{Let } P(x) = \ell x^3 + mx^2 - 4$$

As given that $P(x)$ when divided by $x - 1$ leaves remainder -3, so

$$R = -3$$

$$P(1) = -3$$

$$\ell(1)^3 + m(1)^2 - 4 = -3$$

$$\ell + m - 4 = -3$$

$$\ell + m = 4 - 3$$

$$\ell + m = 1 \dots (i)$$

As given that $P(x)$ when divided by $(x + 2)$ leaves the remainder 12, so

$$R = 12$$

$$P(-2) = 12$$

$$\ell(-2)^3 + m(-2)^2 - 4 = 12$$

$$-8\ell + 4m - 4 = 12$$

$$-8\ell + 4m = 12 + 4$$

$$-8\ell + 4m = 16$$

Dividing by 4

$$-2\ell + m = 4 \dots\dots(ii)$$

Subtracting (ii) from (i)

$$\begin{array}{r} \ell + m = 1 \\ -2\ell + m = 4 \\ + \quad - \quad - \\ \hline 3\ell = -3 \\ \ell = \frac{-3}{3} \\ \ell = -1 \end{array}$$

Putting value of ' ℓ ' in (i)

$$\ell + m = 1$$

$$-1 + m = 1$$

$$m = 1 + 1$$

$$m = 2$$

Q.9 The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible by $x^2 - 5x + 6$. Find the values of a and b

Sol:

$$\text{Let } P(x) = ax^3 - 9x^2 + bx + 3a$$

$$\text{Taking } x^2 - 5x + 6$$

$$= x^2 - 2x - 3x + 6$$

$$= x(x-2) - 3(x-2)$$

$$= (x-2)(x-3)$$

As given that $P(x)$ is exactly divisible by $(x-2)$, so $P(2) = 0$

$$a(2)^3 - 9(2)^2 + b(2) + 3a = 0$$

$$8a - 36 + 2b + 3a = 0$$

$$11a + 2b = 36 \dots\dots(i)$$

As given that $P(x)$ is exactly divisible by $x-3$, so

$$P(3) = 0$$

$$a(3)^3 - 9(3)^2 + b(3) + 3a = 0$$

$$27a - 81 + 3b + 3a = 0$$

$$30a + 3b = 81$$

Dividing by 3

$$10a + b = 27 \dots\dots(ii)$$

Multiplying (ii) by 2 and subtracting (i) from it.

$$20a + 2b = 54$$

$$11a + 2b = 36$$

$$\begin{array}{r} - \quad - \quad - \\ \hline \end{array}$$

$$9a = 18$$

$$a = \frac{18}{9}$$

$$a = 2$$

Putting value of ' a ' in (ii)

$$10a + b = 27$$

$$10(2) + b = 27$$

$$b = 27 - 20$$

$$b = 7$$

Rational Root Theorem

Let

$$a_0x^n + a_1x^{n-1} + \dots\dots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0$$

be a polynomial equation of degree n with integral coefficients. If p/q is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is a factor of the leading coefficient a_0 .

Example

Factorize the polynomial

$$x^3 - 4x^2 + x + 6, \text{ by using Factor}$$

Theorem.

Solution:

$$\text{We have } P(x) = x^3 - 4x^2 + x + 6.$$

Possible factors of the constant term $p = 6$ are $\pm 1, \pm 2, \pm 3$, and ± 6 and of leading coefficient $q = 1$ are ± 1 . Thus the expected zeros (or roots) of $P(x) = 0$ are $\frac{p}{q} = \pm 1, \pm 2, \pm 3$ and ± 6 . If $x = a$ is a zero of $P(x)$, then $(x - a)$ will be a factor.

We use the hit and trial method to find zeros of $P(x)$. Let us try $x = 1$.

$$\begin{aligned}\text{Now } P(1) &= (1)^3 - 4(1)^2 + 1 + 6 \\ &= 1 - 4 + 1 + 6 \\ &= 4 \neq 0\end{aligned}$$

Hence $x = 1$ is not a zero of $P(x)$.

$$\begin{aligned}\text{Again } P(-1) &= (-1)^3 - 4(-1)^2 - 1 + 6 \\ &= -1 - 4 - 1 + 6 = 0\end{aligned}$$

Hence $x = -1$ is a zero of $P(x)$ and therefore,

$$x - (-1) = (x + 1) \text{ is a factor of } P(x).$$

$$\begin{aligned}\text{Now } P(2) &= (2)^3 - 4(2)^2 + 2 + 6 \\ &= 8 - 16 + 2 + 6 = 0 \Rightarrow x = 2 \text{ is a root.}\end{aligned}$$

Hence $(x - 2)$ is also a factor of $P(x)$.

$$\begin{aligned}\text{Similarly } P(3) &= (3)^3 - 4(3)^2 + 3 + 6 \\ &= 27 - 36 + 3 + 6 = 0 \Rightarrow x = 3 \text{ is a zero of } P(x).\end{aligned}$$

Hence $(x - 3)$ is the third factor of $P(x)$.

Thus the factorized form of

$$\begin{aligned}P(x) &= x^3 - 4x^2 + x + 6 \text{ is} \\ &(x + 1)(x - 2)(x - 3).\end{aligned}$$