

# INTRODUCTION TO COORDINATE GEOMETRY

## Define Coordinate Geometry

The study of geometrical shapes in a plane is called plane geometry. Coordinate geometry is the study of geometrical shapes in the Cartesian plane (coordinate plane).

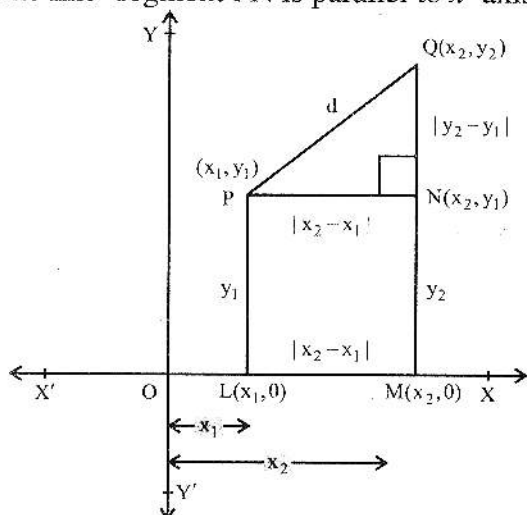
We know that a plane is divided into four quadrants by two perpendicular lines called the axes intersecting at origin. We have also seen that there is one to one correspondence between the points of the plane and the ordered pairs in  $\mathbb{R} \times \mathbb{R}$ .

## Finding Distance between two points

Let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be two points in the coordinate plane where  $d$  is the length of the line segment  $PQ$ . i.e.  $|PQ| = d$ .

The line segments  $MQ$  and  $LP$  parallel to  $y$ -axis meet  $x$ -axis at points  $M$  and  $L$ , respectively with coordinates  $M(x_2, 0)$  and  $L(x_1, 0)$ .

The line-segment  $PN$  is parallel to  $x$ -axis.



In the right triangle  $PNQ$ ,

$$|QN| = |y_2 - y_1| \text{ and } |PN| = |x_2 - x_1|.$$

Using Pythagoras Theorem

$$(\overline{PQ})^2 = (\overline{PN})^2 + (\overline{QN})^2$$

$$\Rightarrow d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\Rightarrow d = \pm \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

since  $d > 0$  always.

## Example

Using the distance formula, find the distance between the points.

(i)  $P(1, 2)$  and  $Q(0, 3)$

(ii)  $S(-1, 3)$  and  $R(3, -2)$

## Solutions

$$\begin{aligned} \text{(i)} |PQ| &= \sqrt{(0-1)^2 + (3-2)^2} \\ &= \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2} \end{aligned}$$

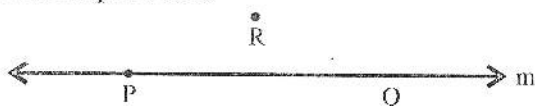
$$\begin{aligned} \text{(ii)} |SR| &= \sqrt{(3-(-1))^2 + (-2-3)^2} \\ &= \sqrt{(3+1)^2 + (-5)^2} = \sqrt{16+25} = \sqrt{41} \end{aligned}$$

## Collinear or Non-collinear Points in the Plane

Two or more than two points which lie on the same straight line are called collinear points with respect to that line; otherwise they are called non-collinear.

Let  $PQ$  be a line, then all the points on line  $m$  are collinear.

In the given figure the points P and Q are collinear with respect to the line  $m$  and the points P and R are not collinear with respect to it.



### Use of Distance Formula to show the Collinearity of Three or more Points in the Plane

Let P, Q and R be three points in the plane. They are called collinear

If  $|PQ| + |QR| = |PR|$ , otherwise they are non-collinear.

#### Example

Using distance formula show that the points.

(i) P(-2,-1), Q(0, 3) and R(1, 5) are collinear.

(ii) The above P,Q,R and S (1,-1) are not collinear

Sol. By using the distance formula, we find

$$|PQ| = \sqrt{(0+2)^2 + (3+1)^2} \\ = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$|QR| = \sqrt{(1-0)^2 + (5-3)^2} \\ = \sqrt{1+4} = \sqrt{5}$$

$$|PR| = \sqrt{(1+2)^2 + (5+1)^2} \\ = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

$$\text{Since } |PQ| + |QR| = 2\sqrt{5} + \sqrt{5}$$

$$= 3\sqrt{5} = |PR|$$

points P, Q, R are collinear.

(ii) The above points P,Q,R and S (1,-1) are not collinear

$$\text{Sol } |PS| = \sqrt{(-2-1)^2 + (-1+1)^2} \\ = \sqrt{(-3)^2 + 0} = 3$$

$$\text{Since } |QS| = \sqrt{(1-0)^2 + (-1-3)^2} \\ = \sqrt{1+16} = \sqrt{17},$$

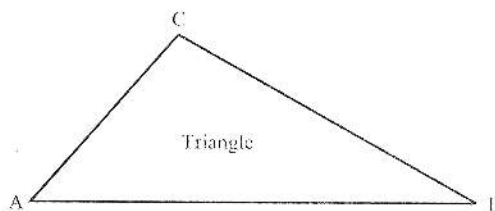
$$\text{and } |PQ| + |QS| \neq |PS|,$$

Therefore the points, P,Q and S are not collinear and hence, the points P, Q, R and S are also not collinear.

### Define Triangle

A closed figure in a plane obtained by joining three non-collinear points is called a triangle.

In the triangle ABC the non-collinear points A, B and C are the three vertices of the triangle ABC. The line segments AB, BC and CA are called sides of the triangle.



### Define Equilateral Triangle

If the lengths of all the three sides of a triangle are same, then the triangle is called an equilateral triangle.

#### Example

The triangle OPQ is an equilateral triangle

since the points O(0,0),  $P\left(\frac{1}{\sqrt{2}}, 0\right)$  and

$Q\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$  are not collinear, where

$$|OP| = \frac{1}{\sqrt{2}}$$

$$|QO| = \sqrt{\left(0 - \frac{1}{2\sqrt{2}}\right)^2 + \left(0 - \frac{\sqrt{3}}{2\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{8} + \frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$

$$|PQ| = \sqrt{\left(\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}} - 0\right)^2}$$

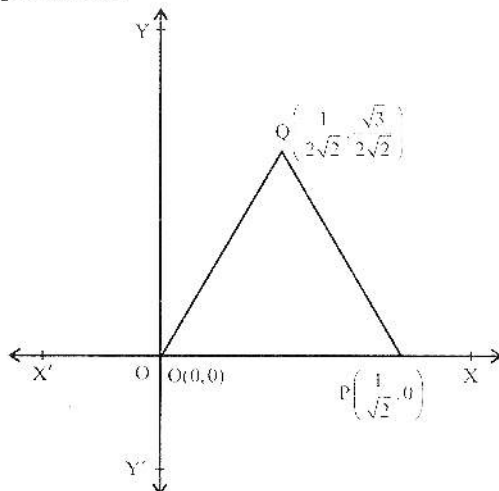
$$= \sqrt{\left(\frac{1-2}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{8} + \frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$

i.e.,  $|OP| = |QO| = |PQ| = \frac{1}{\sqrt{2}}$ , a real number

and the points  $O(0,0)$ ,

$Q\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$  and  $P\left(\frac{1}{\sqrt{2}}, 0\right)$  are not collinear. Hence the triangle  $OPQ$  is equilateral.

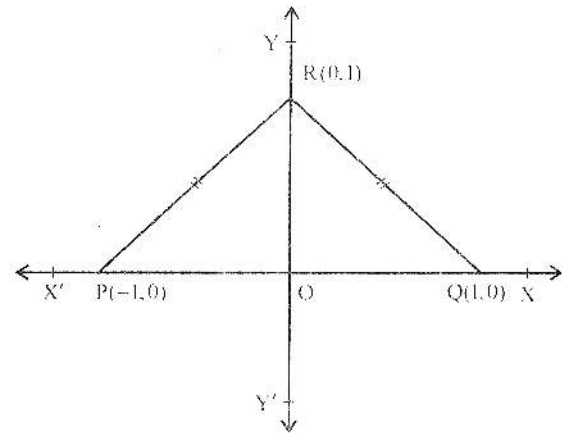


### An Isosceles Triangle

An isosceles triangle  $PQR$  is a triangle which has two of its sides with equal length while the third side has a different length.

### Example

The triangle  $PQR$  is an isosceles triangle as for the non-collinear points  $P(-1,0)$ ,  $Q(1,0)$  and  $R(0,1)$  shown in the following figure.



$$|PQ| = \sqrt{(1 - (-1))^2 + (0 - 0)^2} = \sqrt{(1+1)^2 + 0} = \sqrt{4} = 2$$

$$|QR| = \sqrt{(0 - 1)^2 + (1 - 0)^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$|PR| = \sqrt{(0 - (-1))^2 + (1 - 0)^2} = \sqrt{1+1} = \sqrt{2}$$

Since  $|QR| = |PR| = \sqrt{2}$  and  $|PQ| = 2 \neq \sqrt{2}$  so the non-collinear points  $P$ ,  $Q$ ,  $R$  form an isosceles triangle  $PQR$ .

### Right Angle Triangle

A triangle in which one of the angles has measure equal to  $90^\circ$  is called a right angle triangle.

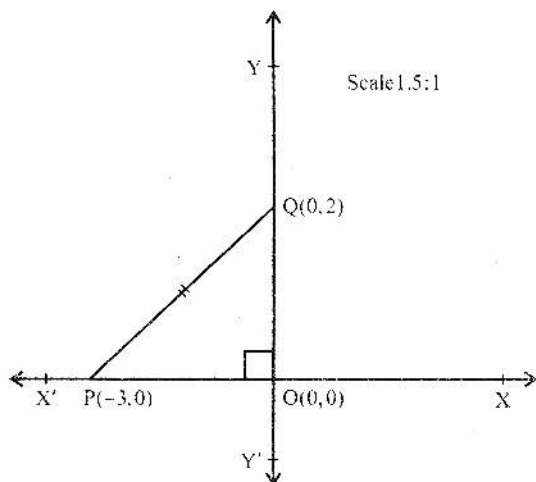
### Example

Let  $O(0,0)$ ,  $P(-3,0)$  and  $Q(0,2)$  be three non-collinear points. Verify that triangle  $OPQ$  is right-angled.

$$|OQ| = \sqrt{(0 - 0)^2 + (2 - 0)^2} = \sqrt{2^2} = 2$$

$$|OP| = \sqrt{(-3)^2 + 0^2} = \sqrt{9} = 3$$

$$|PQ| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$



Now

$$|OQ|^2 + |OP|^2 = (2)^2 + (3)^2 = 13 \text{ and } |PQ|^2 = 13$$

Since

$$|OQ|^2 + |OP|^2 = |PQ|^2, \text{ therefore } \angle POQ = 90^\circ$$

Hence the given non-collinear points form a right triangle.

### Scalene Triangle

A triangle is called a scalene triangle if measures of all the three sides are different.

### Example

Show that the points  $P(1, 2)$ ,  $Q(-2, 1)$  and  $R(2, 1)$  in the plane form a scalene triangle.

### Solution

$$\begin{aligned} |PQ| &= \sqrt{(-2-1)^2 + (1-2)^2} \\ &= \sqrt{(-3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10} \end{aligned}$$

$$\begin{aligned} |QR| &= \sqrt{(2+2)^2 + (1-1)^2} \\ &= \sqrt{4^2 + 0^2} = \sqrt{4^2} = 4 \end{aligned}$$

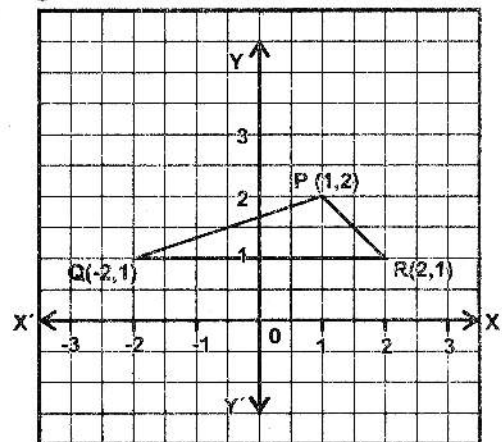
and

$$\begin{aligned} |PR| &= \sqrt{(2-1)^2 + (1-2)^2} \\ &= \sqrt{1^2 + (-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2} \end{aligned}$$

$$\text{Hence } |PQ| = \sqrt{10}, |QR| = 4 \text{ and } |PR| = \sqrt{2}$$

The points  $P$ ,  $Q$  and  $R$  are non-collinear since,  $|PQ| + |QR| > |PR|$

Thus the given points form a scalene triangle.



### Example

If  $A(2, 2)$ ,  $B(2, -2)$ ,  $C(-2, -2)$  and  $D(-2, 2)$  be four non-collinear points in the plane, then verify that they form a square ABCD.

### Solution

$$\begin{aligned} \text{Since } |AB| &= \sqrt{(2-2)^2 + (-2-2)^2} \\ &= \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4 \end{aligned}$$

$$\begin{aligned} |BC| &= \sqrt{(-2-2)^2 + (-2+2)^2} \\ &= \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4 \end{aligned}$$

$$\begin{aligned} |CD| &= \sqrt{(-2-(-2))^2 + (2-(-2))^2} \\ &= \sqrt{(-2+2)^2 + (2+2)^2} \\ &= \sqrt{0+16} = \sqrt{16} = 4 \end{aligned}$$

$$\begin{aligned} |DA| &= \sqrt{(2+2)^2 + (2-2)^2} \\ &= \sqrt{(+4)^2 + 0} = \sqrt{16} = 4 \end{aligned}$$

$$\text{Hence } |AB| = |BC| = |CD| = |DA| = 4$$

$$\text{Also } |AC| = \sqrt{(-2-2)^2 + (-2-2)^2}$$

$$= \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$$

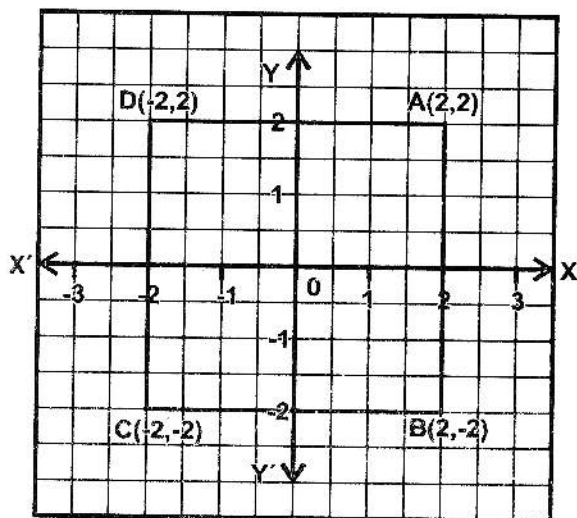
$$\text{Now } |AB|^2 + |BC|^2 = (4)^2 + (4)^2 = 32, \text{ and}$$

$$|AC|^2 = (4\sqrt{2})^2 = 32$$

$$\text{Since } |AB|^2 + |BC|^2 = |AC|^2,$$

therefore  $\angle ABC = 90^\circ$

Hence the given four non-collinear points form a square.



### Define Parallelogram

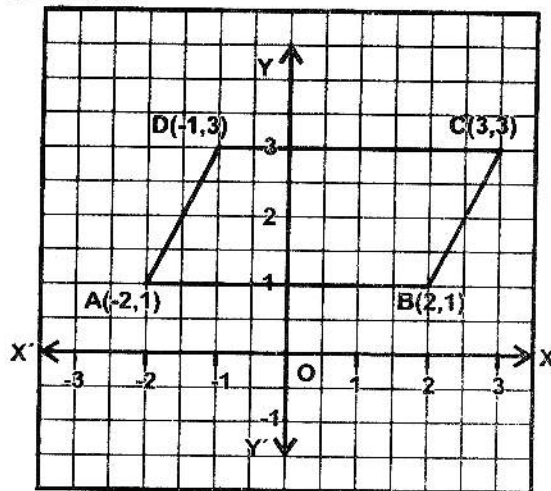
A figure formed by four non-collinear points in the plane is called a parallelogram if

- (i) its opposite sides are of equal length
- (ii) its opposite sides are parallel
- (iii) measure of none of the angles is  $90^\circ$

### Example

Show that the points  $A(-2, 1)$ ,  $B(2, 1)$ ,  $C(3, 3)$  and  $D(-1, 3)$  form a parallelogram.

### Solution:



By distance formula,

$$|AB| = \sqrt{(2+2)^2 + (1-1)^2}$$

$$= \sqrt{4^2 + 0} = \sqrt{16} = 4$$

$$|CD| = \sqrt{(3+1)^2 + (3-3)^2}$$

$$= \sqrt{4^2 + 0} = \sqrt{16} = 4$$

$$|AD| = \sqrt{(-1+2)^2 + (3-1)^2}$$

$$= \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$|BC| = \sqrt{(3-2)^2 + (3-1)^2}$$

$$= \sqrt{1^2 + 2^2} = \sqrt{5}$$

Since

$$|AB| = |CD| = 4 \text{ and } |AD| = |BC| = \sqrt{5}$$

So opposite sides of the quadrilateral ABCD are equal.

$$\text{Also } |AC| = \sqrt{(3+2)^2 + (3-1)^2}$$

$$= \sqrt{(5)^2 + 2^2} = \sqrt{25+4} = \sqrt{29}$$

Now

$$|AB|^2 + |BC|^2 = 16 + 5 = 21 \text{ and } |AC|^2 = 29$$

Since in triangle

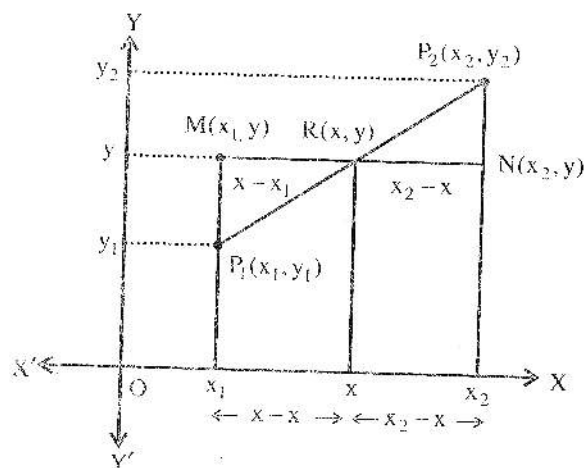
$$ABC, |AB|^2 + |BC|^2 \neq |AC|^2$$

Therefore measure of angle at B  $\neq 90^\circ$

Hence the given points form a parallelogram.

### Recognition of the Mid-Point Formula for any two Points in the Plane

Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be any two points in the plane and  $R(x, y)$  be a mid-point of points  $P_1$  and  $P_2$  on the line-segment  $P_1P_2$  as shown in the figure below.



If line-segment MN, parallel to x-axis, has its mid-point  $R(x, y)$ ,

then,  $x_2 - x = x - x_1$

$$\Rightarrow 2x = x_1 + x_2 \Rightarrow x = \frac{x_1 + x_2}{2}$$

$$\text{Similarly, } y = \frac{y_1 + y_2}{2}$$

Thus the point  $R(x, y) = R\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$  is the mid-point of the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ .

### Example

Find the mid-point of the line segment joining  $A(2, 5)$  and  $B(-1, 1)$ .

### Solution

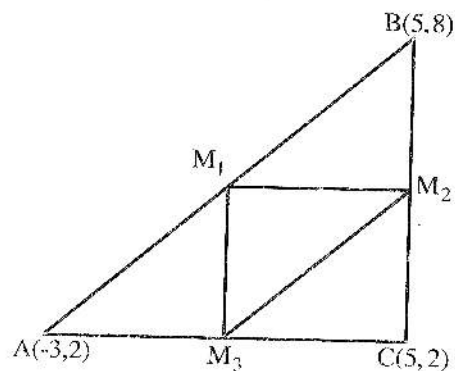
If  $R(x, y)$  is the desired mid-point then,

$$x = \frac{2 + (-1)}{2} = \frac{1}{2} \text{ and } y = \frac{5 + 1}{2} = \frac{6}{2} = 3$$

$$\text{Hence } R(x, y) = R\left(\frac{1}{2}, 3\right)$$

### Example

Let ABC be a triangle as shown below. If  $M_1, M_2$  and  $M_3$  are the middle points of the line-segments AB, BC and CA respectively, find the coordinates of  $M_1, M_2$  and  $M_3$ . Also determine the type of the triangle  $M_1M_2M_3$ .



### Solution

$$\text{Midpoint of } AB = M_1\left(\frac{-3 + 5}{2}, \frac{2 + 8}{2}\right) = M_1(1, 5)$$

$$\text{Midpoint of } BC = M_2\left(\frac{5 + 5}{2}, \frac{8 + 2}{2}\right) = M_2(5, 5)$$

and Mid-point of

$$AC = M_3\left(\frac{5 - 3}{2}, \frac{2 + 2}{2}\right) = M_3(1, 2)$$

The triangle  $M_1M_2M_3$  has sides with length,

$$|M_1M_2| = \sqrt{(5-1)^2 + (5-5)^2} \\ = \sqrt{4^2 + 0} = 4 \dots (i)$$

$$|M_2M_3| = \sqrt{(1-5)^2 + (2-5)^2} \\ = \sqrt{(-4)^2 + (-3)^2} \\ = \sqrt{16+9} = \sqrt{25} = 5 \dots (ii)$$

and  $|M_1M_3| = \sqrt{(1-1)^2 + (2-5)^2} \\ = \sqrt{0^2 + (-3)^2} = 3 \dots (iii)$

All the lengths of the three sides are different. Hence the triangle  $M_1M_2M_3$  is a Scalene triangle.

### Example

Let  $O(0,0)$ ,  $A(3,0)$  and  $B(3,5)$  be three points in the plane. If  $M_1$  is the mid point of  $AB$  and  $M_2$  of  $OB$ , then show that  $|M_1M_2| = \frac{1}{2}|OA|$ .

### Solution

By the distance formula the distance

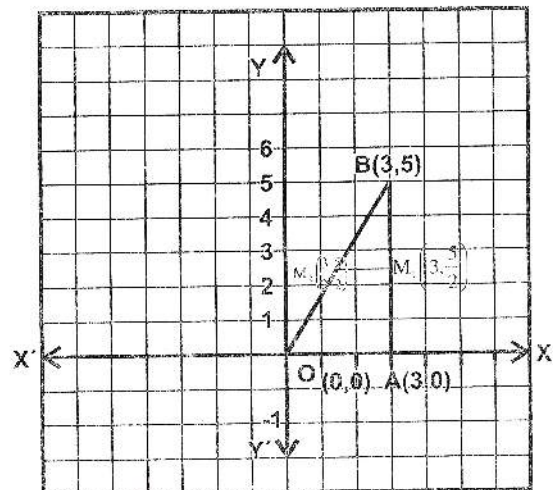
$$|OA| = \sqrt{(3-0)^2 + (0-0)^2} = \sqrt{3^2} = 3$$

The mid-point of  $AB$  is:

$$M_1 = M_1\left(\frac{3+3}{2}, \frac{5+0}{2}\right) = \left(3, \frac{5}{2}\right)$$

The mid-point of  $OB$  is:

$$M_2 = M_2\left(\frac{3+0}{2}, \frac{5+0}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$$



Hence

$$|M_1M_2| = \sqrt{\left(\frac{3}{2}-3\right)^2 + \left(\frac{5}{2}-\frac{5}{2}\right)^2} \\ = \sqrt{\left(\frac{-3}{2}\right)^2 + 0} = \sqrt{\frac{9}{4} + 0} = \frac{3}{2} \\ = \frac{1}{2}|OA|$$

### Note

Let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be any two points and their midpoint be:

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right). \text{ Then } M$$

- (i) is at equal distance from  $P$  and  $Q$   
i.e.,  $|PM| = |MQ|$
- (ii) is an interior point of the line segment  $PQ$ .
- (iii) every point  $R$  in the plane at equal distance from  $P$  and  $Q$  is not their mid point. For example, the point  $R(0,1)$  is at equal distance from  $P(-3, 0)$  and  $Q(3, 0)$  but is not their mid-point.

$$\text{i.e., } |RQ| = \sqrt{(0-3)^2 + (1-0)^2}$$

$$= \sqrt{(-3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$|RP| = \sqrt{(0+3)^2 + (1-0)^2}$$

$$= \sqrt{3^2 + 1^2} = \sqrt{10}$$

And midpoint of P(-3,0) and Q(3, 0) is

$$\text{Where } x = \frac{-3+3}{2} = 0$$

$$y = \frac{0+0}{2} = 0$$

The point (0, 1)  $\neq$  (0, 0).

(iv) There is a unique midpoint of any two points in the plane.

## Exercise 9.1

**Q1.** Find the distance between the following pairs of points

a) A(9, 2), B(7, 2)

$$\begin{aligned} \text{Sol. } |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(7-9)^2 + (2-2)^2} \\ &= \sqrt{(-2)^2 + (0)^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

b) A(2, -6), B(3, -6)

$$\begin{aligned} \text{Sol. } |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3-2)^2 + (-6+6)^2} \\ &= \sqrt{(1)^2 + (0)^2} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

c) A(-8, 1), B(6, 1)

$$\begin{aligned} \text{Sol. } |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6+8)^2 + (1-1)^2} \\ &= \sqrt{(14)^2 + (0)^2} \end{aligned}$$

$$|AB| = 14$$

d) A(-4,  $\sqrt{2}$ ), B(-4, -3)

$$\begin{aligned} \text{Sol. } |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-4+4)^2 + (-3-\sqrt{2})^2} \\ &= \sqrt{(0)^2 + (-3-\sqrt{2})^2} \\ &= \sqrt{(-3-\sqrt{2})^2} \\ &= (3+\sqrt{2})^2 \\ &= 3+\sqrt{2} \end{aligned}$$

(e) A(3, -11), B(3, -4)

$$\begin{aligned} \text{Sol. } |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3-3)^2 + (-4-(-11))^2} \\ &= \sqrt{(0)^2 + (7)^2} = \sqrt{(7)^2} = 7 \end{aligned}$$

(f) A(0,0) B(0,-5)

$$\begin{aligned} |AB| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \text{Sol. } &= \sqrt{(0-0)^2 + (-5-0)^2} \\ &= \sqrt{0+(-5)^2} = \sqrt{(5)^2} = 5 \end{aligned}$$

**Q2.** Let P be the point on x-axis with x-coordinate a and Q be the point on y-axis with y-coordinate b, as given below. Find distance between P and Q.

i)  $a = 9, b = 7$

$$|PQ| = \sqrt{(9)^2 + (7)^2} = \sqrt{81 + 49} = \sqrt{130}$$

ii)  $a = 2, b = 3$

$$|PQ| = \sqrt{(2)^2 + (3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

iii)  $a = -8, b = 6$

iv)  $|PQ| = \sqrt{(-8)^2 + (6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$   
 $a = -2, b = -3$

$$|PQ| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

v)  $a = \sqrt{2}, b = 1$

$$|PQ| = \sqrt{(\sqrt{2})^2 + (1)^2} = \sqrt{2 + 1} = \sqrt{3}$$

vi)  $a = -9, b = -4$

$$|PQ| = \sqrt{(-9)^2 + (-4)^2} = \sqrt{81 + 16} = \sqrt{97}$$